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RESEARCH ARTICLE

Generalized majority rules: utilitarian welfare in large but finite populations

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Abstract
Generalized majority rules are electoral rules in which an alternative needs to obtain a fixed percentage (not necessarily 50%) of all votes in order to win. While Krishna and Morgan (Am Econ J Microeconom 7:339–375, 2015) demonstrate that simple majority maximizes expected utilitarian welfare for limiting populations regardless of the prior support for the alternatives, this paper finds that, when the prior support is known, a continuum of voting rules also achieves the same welfare. Moreover, as the population approaches the limit, every voting rule eventually becomes welfare inferior to picking the ex-ante majority without an election. Examining the properties of these optimal rules allows us to generalize the relationship between voter participation and welfare beyond the symmetric case.

Keywords Supermajority · Qualified majority · Costly voting · Voter turnout · Compulsory voting · Poisson game

JEL Classification D72 · C72

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1 Introduction

Majority rules with thresholds different from one-half are not unusual in two-alternative elections. For example, the Constitution of the USA can be amended either by a two-third vote in each house or a proposal called by two-third of the states and subsequently ratified by three-quarters of the states. The state of New York requires a three-fifth majority to pass most tax increases. In corporate finance, a "supermajority amendment" requires significantly more than half of the shareholders to approve a merger or other major actions of a company.

Even if one contends that simple majority rule is "the" majority rule to be considered, studying supermajority rules is still useful for two purposes. First, if a fraction of the population is always voting (i.e., if they derive net utility from voting), and the votes from this sub-population are not evenly split, then the election from the point of view of the remaining citizens would essentially be one with an asymmetric threshold.[1] For this purpose, a model that predicts voting behavior under supermajority rules is called for. Second, the optimality of simple majority rule is evaluated in light of its alternatives, i.e., qualified majorities.[2] Thus, a game theoretic model comparing welfare under different supermajority rules could potentially justify the use of simple majority.

A generalized majority rule (GMR) in a two alternative election consists of two components. First, an electoral threshold is specifying the minimum share of the votes that one of the alternatives must exceed in order to win; second, a tie-breaking rule specifying this alternative's winning probability in case its share of the votes is exactly equal to the threshold. The set of GMRs includes, among others, simple majority rule, supermajority rules and deterministic rules (selecting one of the alternatives regardless of the voting outcome). While GMRs are not the only possible two-alternative voting rules, Grüner and Tröger (2019) show that, in a two-alternative social choice setting, any ex-ante expected utility maximizing anonymous social decision mechanism with voluntary participation is outcome equivalent to a linear voting rule. As generalized majority rules form a common class—though not the only class—of linear voting rules, our restriction to GMRs is well-justified in light of this broader context.

In the above spirit, Krishna and Morgan (2015) generalizes the extant literature on costly voting under symmetric voting rules[3] by studying the welfare implications of supermajority rules as the population goes to infinity. Importantly, they show that *if the social planner is agnostic about the prior support of the two alternatives*, then simple majority is the only rule implementing the ex-ante utility maximizing alternative (the "utilitarian choice", as they call it) with probability 1 at the limit. This is in spite of—and indeed because of—the "underdog effect" (i.e., the minority votes more often than the majority, see Levine and Palfrey 2007), as they observe that the underdog effect can never tip the relative turnout over the one-half threshold. While axiomatic foundations for simple majority rule have been provided assuming exogenous participation (e.g.,

[1] We thank Sourav Bhattacharya for suggesting this interpretation to us.

[2] Throughout the paper, we will use indifferently the terms supermajority and qualified majority.

[3] For examples, Palfrey and Rosenthal (1983, 1985); Ledyard (1981, 1984); Taylor and Yildirim (2010); Herrera et al. (2014).

May 1952; Dasgupta and Maskin 2010), Krishna and Morgan's (2015) result provides a positive justification for the prevalence of simple majority rule in environments where voters' participation is endogenous.

Krishna and Morgan's (2015) result requires no knowledge of the prior support of the two alternatives. But what if the prior support is known? This is our point of departure. We show that, if prior supports are asymmetric then welfare is maximized by a continuum of GMRs at the limit. This continuum ranges from selecting the ex-ante majority regardless of the voting outcome to a critical threshold lying between one-half and the ex-ante support of the majority. Such threshold handicaps the majority party, as it requires more than half of the votes to win, but does so only partly, as the threshold is lower than its prior support. Essentially, under these rules (which include simple majority), the utilitarian choice wins with probability 1 at the limit. Meanwhile, because of the "paradox of voting" (i.e., turnout rate goes to zero as the electorate grows without bound, see Downs 1957; Palfrey and Rosenthal 1985), the net benefit of voting [4] is zero across all these rules at the limit. Thus, at the limit, all these voting rules are welfare equivalent.

However, the same intuition does not apply *as we approach the limit*. The reason is that, as one approaches the limit, turnout remains strictly greater than zero and, consequently, there is a positive, albeit vanishing, probability of *not* implementing the utilitarian choice (i.e., of making a "mistake", from a social perspective). Meanwhile, there is a positive, though again vanishing, net benefit from voting. Thus, for *large but finite populations* it is essential to weigh the net benefits from voting against the cost of making mistakes. Taking this into account, it is no longer a priori clear that the optimal rule at the limit should also be optimal along the sequence approaching the limit.

Indeed, our main result establishes that, *in contrast to what happens at the limit*, as the population approaches the limit, each non-deterministic voting rule will eventually become welfare inferior to the deterministic rule that picks the ex-ante majoritarian option without holding an election, provided that a unique ex-ante majoritarian option exists. This is because, as the population increases, the cost of making the wrong choice diminishes at a lower rate than the benefit of voting does. Thus, for large but finite populations, the cost of making a mistake through voluntary voting is larger than the net benefit from voting foregone by simply selecting the ex-ante majoritarian choice.

Prima facie, this finding may appear self-evident: the (ex-ante) majority wins the election, while society incurs no voting costs. As it turns out, this is not that obvious. To understand why, it is helpful to read our result in light of Börgers's (2004) seminal contribution on the welfare analysis of voluntary voting by simple majority. Under the assumption of a symmetrically split electorate, Börgers poses the following question: in the absence of an ex-ante majority, why not randomly choose the winner, thus saving voting costs? But Börgers shows this is not the case. First note that, by symmetry, either side is equally likely to win under both regimes. Hence, those who choose to abstain when voting is voluntary will be equally well off if the winner is picked by flipping a fair coin. However, applying a revealed preferences argument, those who choose to vote

[4] We refer by this to the marginal benefit from voting net of the voting cost.

under simple majority rule must (strictly) prefer voting to abstaining. It follows that these agents must be (strictly) worse off under random decision making, resulting in a lower ex-ante expected social welfare. In light of Börgers's intuition, it is surprising that our result suggests exactly the opposite: that no voting is socially optimal. Crucially, the symmetry assumption in Börgers's model implies the impossibility of "mistakes"—no alternative can be a wrong choice. Yet once we introduce asymmetric support and hence the possibility of a wrong choice, the cost of mistakes is of a larger magnitude than the net voting benefits.

This result is significant in two ways. First, it exposes a (lower hemi-)discontinuity of the optimal voting rule at the limit as the population increases. If we use the approaching sequence as a selection among the continuum of optimal voting rules at the limit, our result selects a "tyranny of the majority" regime, rather than simple majority. Granted, knowing the ex-ante majority is necessary to implement our optimal rule, while that is not required to implement simple majority rule. In this sense, if Krishna and Morgan's (2015) result justifies the use of simple majority in referenda (to discover the utilitarian choice), our result can be interpreted as a warning against direct democracy (when the majoritarian opinions on routine policies are known).

Second, our result reveals a striking dichotomy between welfare and turnout: the former is maximized when no (real) election is held, that is when the latter is minimized. Such a stark disjunction is surprising insomuch as the literature on costly voting has instead identified a coincidence between welfare and (voluntary) turnout. This correspondence is already perceivable in the discussion of Börgers's (2004) welfare analysis above. Subsequently, the coincidence between turnout and welfare has been suggested by papers studying turnout and welfare across different electoral rules. Both Kartal (2015) and Faravelli and Sanchez-Pages (2015) establish that in evenly split electorates simple majority rule maximizes both welfare and turnout.

To make sense of this seeming inconsistency, we study the maximization problem of expected ex-ante voting benefits and demonstrate that, with known prior support, voting benefits are maximized by the GMR coinciding with the critical threshold mentioned earlier. As voter participation is positively related to the benefit from voting, a GMR at the critical threshold also maximizes turnout. This threshold, which generally lies between one-half and the majority's prior support, is equal to one-half if and only if the population is symmetrically split. At the same time, as previously noted, if the two alternatives are ex-ante symmetric the expected benefit from abstaining is independent of the voting rule adopted. Thus, welfare for an evenly split electorate will be determined solely by the net benefit of voting. It follows that simple majority rule maximizes both turnout and expected payoff if and only if the prior supports are equal, a coincidence which is purely an artefact of symmetry.[5]

Following along the line of welfare analysis, we compare voluntary voting under GMRs with two antithetic regimes: compulsory voting and random decision making. Börgers (2004) has shown that, for an evenly split population, voluntary voting under simple majority is welfare superior to both compulsory voting and random decision

[5] Börgers (2004) requires exact symmetry in the support. Krasa and Polborn (2009); Faravelli and Sanchez-Pages (2015); Kartal (2015) give conclusive welfare comparisons only for an "almost symmetric" support and "almost unanimous" society.

making.[6] We obtain a similar comparison between voluntary voting and compulsory voting: voluntary voting is welfare superior because compulsory voting is too costly, a factor unaffected by the asymmetry of either the prior support or the voting rule. The comparison between voluntary voting and random decision warrants some discussions, as our main result has already indicated that voluntary voting is welfare inferior to selecting the ex-ante majoritarian choice without an election. However, random decision making is different from selecting the ex-ante majoritarian choice—the former has a (non-vanishing) probability of making a wrong choice. It is exactly for this reason that random decision making is not only welfare inferior to the outright selection of the ex-ante majoritarian choice, but also to any voting rule that selects the utilitarian choice with probability 1 at the limit.

With regard to the selection of the "correct" choice, our work is related to the vast literature on the Condorcet jury theorem, i.e., the notion that majorities are more likely than individuals to select the "correct" alternative in a common value setting with imperfect information. Of the many versions of the Condorcet jury problem, the most relevant for our paper are those that allow strategic voting (Austen-Smith and Banks 1996; Koriyama and Szentes 2009; Krishna and Morgan 2011, 2012). In particular, Feddersen and Pesendorfer (1998) consider majority rules with different thresholds, showing how any non-unanimous rule results in a lower probability of error than unanimity. Unlike this literature, though, we focus on a private value setting. Our primary interest is the social welfare generated by the voting process, instead of the ability to aggregate information.

As a final remark, we should note that several studies have analyzed different properties of supermajorities, such as their ability to implement only Pareto improvements (Buchanan and Tullock 1962); their stability against Condorcet cycles (Caplin and Nalebuff 1988); their role in relation to self-stable constitutions (Barberà and Jackson 2004); their function as commitment devices or options against dynamic inconsistency (Messner and Polborn 2004, 2012); their role in protecting citizens from unrepresentative legislators (Graham and Bernhardt 2015), to name a few. All of these papers are concerned with a particular rationale for supermajorities and each one of them adopts a specific setting highlighting the rationale of interest. We, instead, wish to consider the canonical model of strategic voting and generalize the notion of majority rule, thereby unravelling the connection between voting incentives, participation and welfare.

2 Model

2.1 Set up

Two parties (or alternatives), A and B, are running for election. The generic party will be denoted as P. The number of citizens is a Poisson random variable with mean n. Each citizen has, independently, the same ex-ante probability $\alpha(A) \in [1/2, 1)$ of

[6] Simple majority is important here insofar it is a symmetric voting rule. See Faravelli et al. (2016).

preferring party A to B and probability $\alpha(B) = 1 - \alpha(A)$ of preferring B to A.[7] Citizens choose simultaneously to vote for (the candidate of) party A, party B or to abstain. If citizen i votes (instead of abstaining), she bears a private voting cost c_i. The voting costs of citizens are identically and independently drawn from the support [0, 1] according to the cumulative distribution function F with a density function f, where $f(c) > 0$ for all $c \in (0, 1)$ (but not necessarily at the end points).[8] The following assumption will be maintained throughout this paper.[9]

Assumption 1 *For any real number $x > 0$,*

$$\lim_{\varepsilon \downarrow 0} \frac{F(\varepsilon x)}{F(\varepsilon)} \in (0, \infty) \text{ and is continuous in } x.$$

Each citizen's party preference and voting cost are private information, but their distributions are commonly known.

A GMR is described by two parameters: a threshold θ, which is a rational number in [0, 1],[10] and a tie-breaking randomization probability $\rho \in (0, 1)$.

Suppose n_A and n_B are the numbers of votes received by A and B, a (θ, ρ)-rule governs

$$\Pr[A \text{ wins} \mid n_A, n_B] = \begin{cases} 1 & \text{if } n_A > \theta(n_A + n_B) \\ \rho & \text{if } n_A = \theta(n_A + n_B) \\ 0 & \text{if } n_A < \theta(n_A + n_B) \end{cases} \text{; and}$$
$$\Pr[B \text{ wins} \mid n_A, n_B] = 1 - \Pr[A \text{ wins} \mid n_A, n_B].$$

A (1/2, 1/2)-rule is simple majority rule. While we will restrict our attention to rules where $\rho \in (0, 1)$, we will sometimes consider the extreme case of an A-deterministic (a (0, 1)-rule). Similarly, a (1, 0)-rule is a B-deterministic rule. Some statements in this paper depend on θ but not ρ. In those cases, we may speak of a θ-rule, which refers to a voting rule with threshold θ and an arbitrary $\rho \in (0, 1)$.

A citizen gets a payoff of 1 if her preferred party wins and 0 if her preferred party loses. Each voting citizen pays her voting cost.

2.2 Equilibrium

The above defines a Poisson game (Myerson 1998). A pure strategy function for a citizen is a measurable function assigning an action (vote for A, for B or abstain) for

[7] We name the parties so that A is always the majority.

[8] We can allow the support of voting cost to be some $C \subseteq \mathbb{R}_+$ with $\inf C = 0$. However, since the highest possible benefit from voting is 1, citizens with voting costs above 1 will never vote.

[9] This assumption is satisfied by any distribution where $f(0) > 0$ and is bounded, the uniform distribution, beta distributions, power function distributions (i.e., $F(c) = c^\gamma$ for $\gamma > 0$) and triangular distributions with the lower bound at 0.

[10] If one insists on having an irrational θ, our analysis can go through with a sequence of rational θ_k's converging to the desired irrational θ.

each party preference and voting cost realization combination in $\{A, B\} \times [0, 1]$. As in standard Poisson games, we will only consider the case in which all citizens use the same pure strategy function.

For each party preference and voting cost realization, voting for the less preferred party is always strictly worse than abstaining regardless of the number of votes cast for the two parties.[11] In addition, the net marginal benefit of voting for one's preferred party (i.e., the payoff from voting for one's preferred party over the payoff from abstaining, minus the voting cost) is strictly decreasing in the voting cost. Therefore, given the expected number of citizens n, any optimal strategy can be described by a pair of cutoff voting costs, $c_n(A)$ and $c_n(B)$, such that a citizen who prefers party P votes for P for all voting costs $c < c_n(P)$ and abstains for all $c > c_n(P)$.[12]

For most of this paper, it would be convenient to work with the probabilities of voting associated with the cutoff costs. Given n, if $c_n(P)$ is the cutoff cost for party P, the probability that a P-supporter votes will be denoted as $p_n(P) = F(c_n(P))$. A voting probability profile is given by $p_n = (p_n(A), p_n(B))$.

The expected gross (i.e., without subtracting the voting cost) marginal benefit from voting (over abstaining) is the expected pivotal benefit. With a GMR, there are three ways in which a vote for A can be pivotal:

Type I: Changing from a tie to a win for party A;
Type II: Changing from a win for party B to a tie; and
Type III: Changing from a win for party B to a win for party A.

Possible pivots by a B-vote can be similarly defined. Figure 1 illustrates these three types of pivotal events for $\theta = 5/8$.

A Type III pivot cannot happen under a simple majority rule. Nonetheless, this can be an important type of pivot for other majority rules. As illustrated in Fig. 1, the number of possible Type III pivot profiles are different across the parties. When $\theta > 1/2$, there are more Type III pivot profiles for a B vote than for an A vote. The reverse is true if $\theta < 1/2$.

Write $U_n(P)$ (which depends on the voting probabilities) as the expected gross marginal benefit from voting for a P-supporter. An equilibrium of a voting game with n expected citizens can be described by a voting profile p_n such that

$$F\left(U_n(P)\right) = p_n(P) \quad \text{for } P = A, B.$$

For each finite n, an equilibrium of a voting game exists by Myerson (2000, Theorem 0).

Fixing a (θ, ρ)-rule, let p_n be an equilibrium voting probability profile of a voting game with n expected citizens. Since $p_n \in [0, 1]^2$, we can pick an index set $\mathcal{I}$ for n such that $\{p_n\}_{n\in\mathcal{I}}$ converges. This qualification on the passing to an appropriate index set will be omitted henceforth.

Given a voting probability profile p_n, let $t_n = \sum_{P=A,B} \alpha(P)p_n(P)$ be the expected aggregate turnout rate in the voting game with n expected citizens.

[11] Except when $c = 0$, which is a zero probability event.

[12] We will not specify a P-aligned citizen's choice when her voting cost is exactly $c_n(P)$ since this is a zero probability event.

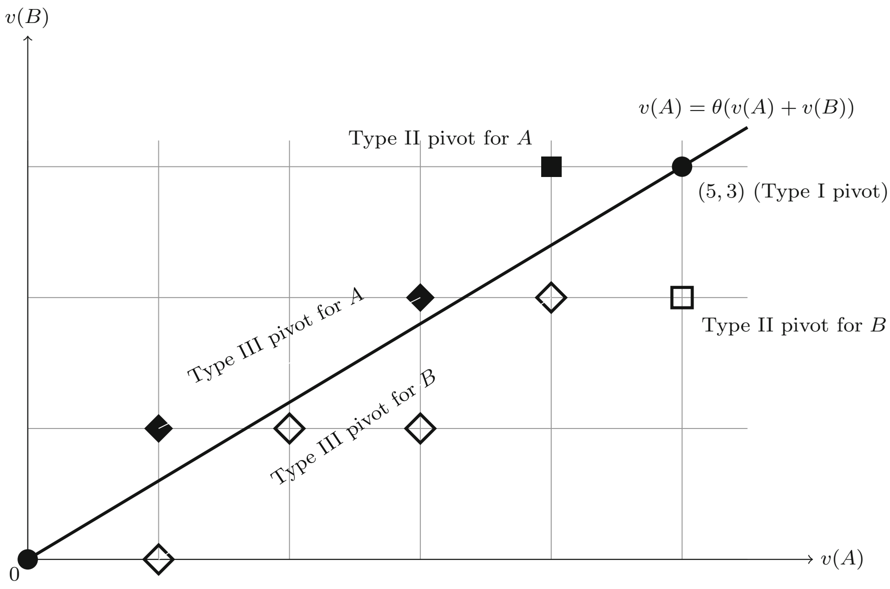

Fig. 1 Graphical Illustration of Pivotal Events ($\theta = 5/8$)

3 Preliminaries

In this section, we re-cap a few limiting equilibrium properties of GMRs from Krishna and Morgan (2015) that we will use later. Interested readers should consult Krishna and Morgan (2015) for discussions and proofs.

Lemma 1 *For any non-deterministic* (θ, ρ)*-rule (i.e.,* $(\theta, \rho) \neq (0, 1)$ *or* $(1, 0)$*), the expected number of citizens who vote in equilibrium,* $t_n n$*, goes to infinity as n does.*

While Lemma 1 is a statement about the absolute number of votes cast, our subsequent analysis will be built upon turnout rates, i.e., the proportion of the population that turns out to vote.

Proposition 1 (Paradox of Voting) *Suppose* $\rho \in (0, 1)$*. For any* (θ, ρ)*-rule, the limiting equilibrium turnout rate is zero. That is,* $\lim_n p_n(P) = 0$ *for each party* P.

Notwithstanding the persistence of the paradox of voting across all generalized majority rules, the proportion of voters for the two parties will decline at different rates as the population increases, thus determining different winning probabilities for the two sides.

Proposition 2 (Krishna and Morgan 2015, Theorem 2) *Given* $\alpha(A) \geq 1/2$*, there exists a unique* $\theta^*(\alpha) \in [1/2, \alpha(A)]$ *such that*

$$\lim_{n\to\infty} \Pr[A wins \mid n] = \begin{cases} 1 & \text{if } \theta < \theta^*(\alpha) \\ 1/2 & \text{if } \theta = \theta^*(\alpha) \\ 0 & \text{if } \theta > \theta^*(\alpha) \end{cases}.$$

In addition, $\theta^*(\alpha) = 1/2$ *if only if* $\alpha(A) = 1/2$.[13]

It should be noted that the probability of winning at the θ^*-rule is $1/2$, rather than ρ, since the randomness is coming from the uncertainty over the population size, preferences and voting cost, rather than from the tie-breaking process. Indeed, the probability of a tie goes to zero as the population grows to infinity, which explains why θ^* and the limiting winning probabilities are independent of ρ.

Two remarks on Proposition 2 are due. First, regardless of the (ex-ante) margin enjoyed by the majority, $\theta^* \geq 1/2$ and strictly so in case of a strict majority. Second, θ^* is less than $\alpha(A)$. In other words, for instance, a 2/3-majority will lose with probability 1 when the electoral threshold is 2/3. This property is caused by the "underdog effect", a well-known feature of simple majority rule (Levine and Palfrey 2007). This has been noted by Krishna and Morgan (citeyearKrishnaMorgan2012b, Proposition 7).

Proposition 3 (Underdog Effect) *Suppose* $\alpha(A) \geq 1/2$. *For any* $\theta \in [1/2, \alpha(A)]$, *the minority votes more frequently than the majority at a* θ*-rule. That is,*

$$\lim_{n\to\infty} \frac{p_n(A)}{p_n(B)} \leq 1.$$

At any $\theta \in [1/2, \alpha(A)]$, the underdog effect may be called "partial" (c.f., Herrera et al. 2014) meaning that the majority still receives more than half of the total votes in expectation. Observing this, Krishna and Morgan (2015) proves the following theorem:

Theorem (Krishna and Morgan 2015, Theorem 1) *A* (θ, ρ)*-rule selects the utilitarian outcome with probability 1 at the limit for each and every* $\alpha(A) \in (0, 1)$ *if and only if* $\theta = 1/2$.[14]

Due to the paradox of voting (Proposition 1), a citizen's limiting payoff is given by the payoff from abstaining, which is simply the probability that her preferred option is elected. Thus the above theorem also implies that the ex-ante expected limiting social payoff is maximized under simple majority.

4 Optimal rules under known support

4.1 Expected limiting payoff maximization

The bite of Krishna and Morgan's (2015) Theorem 1 lies in the "belief-free" requirement that one rule needs to fit all possible values of α. In other words, if a social planner wishes to maximize the expected limiting social welfare but does not know $\alpha(A)$, she can adopt simple majority, regardless of her beliefs about the preference distribution.

[13] There is no guarantee that $\theta^*(\alpha)$ is rational, a requirement for an electoral threshold. For practical purposes, however, one can approximate θ^* by a rational number however close one desires.

[14] For this theorem, $\alpha(A) \in (0, 1)$ (rather than $[1/2, 1)$ as in the rest of the paper) to reflect the ignorance about the identity of the majority party.

But what if $\alpha(A)$ is known? Are there other voting rules that also maximizes the expected limiting social welfare? The answer is yes, and the reason is rather straightforward. As noted above, at the limit, a citizen's payoff is the probability that her preferred option is elected. The representative citizen's limiting ex-ante expected payoff from abstaining (given n) is therefore

$$\lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P)\,\Pr\left[P\text{wins} \mid n\right]. \tag{1}$$

If $\alpha(A) = 1/2$ the above expression is equal to 1/2 independently of what voting rule is applied. If instead $\alpha(A) > 1/2$ (without loss of generality), Expression (1) is maximized as long as the probability that A wins goes to 1 as n goes to infinity. Applying Proposition 2, we have the following theorem:

Theorem 1 (Expected Limiting Payoff Maximization) *If $\alpha(A) > 1/2$, any (θ, ρ)-rule with $\theta < \theta^*$ maximizes the expected limiting social payoff.*

Since $\theta^* > 1/2$ whenever $\alpha(A) > 1/2$, the set of optimal voting rules at the limit always include simple majority, which is consistent with Krishna and Morgan's (2015 Theorem 1).

4.2 Expected payoff maximization for large *n*

Theorem 1 implies that a continuum of GMRs maximizes the limiting expected payoff by implementing the majoritarian choice with probability 1 in the limit. However, expected payoffs under these rules may converge to the limit at different rates. Hence, we ask what voting rule would be optimal for large, but finite n.

Given a (θ, ρ)-rule, let $\overline{U}_n(P)$ be the expected payoff for a P-supporter who abstains. Unlike the limiting case, fixing n, the ex-ante expected payoff to a representative citizen is

$$\sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n(P) + p_n(P)U_n(P) - \int_0^{U_n(P)} c\,\mathrm{d}F(c)\right].$$

What voting rule maximizes the above expression for large, but finite n? The answer is stark.

Theorem 2 (Expected Payoff Maximization for large n) *Suppose $\alpha(A) > 1/2$. For each (θ, ρ)-rule, there exists an N such that this rule yields a strictly lower ex-ante expected payoff than an A-deterministic rule (i.e., the $(0, 1)$-rule) for all $n > N$.*

Theorem 2 states as the population approaches the limit, each non-deterministic voting rule will eventually become welfare inferior to a voting rule which guarantees that the ex-ante majority wins the election with certainty. Such a voting rule implies that, in equilibrium, nobody votes and, therefore, is by all means equivalent to not holding an election, but simply picking the ex-ante majority and declaring it the winner.

One may be tempted to think that this is a trivial result: whenever the population is unevenly split and the ex-ante majority is known, isn't it evidently better to pick the majority while saving the costs of voting?

In order to understand why this is not the case, it is useful to consider one of the main results in Börgers (2004). Assume an ex-ante evenly split population. Symmetry implies that either party is equally likely to win the election in equilibrium. Hence, Börgers asks the following question: wouldn't it be better to randomly pick one of the two sides and save the costs of voting? The answer is no and the reason is the following: While the members of either party who abstain from voting in the election are indeed indifferent between holding an election or randomly picking a winner, this is not true for those who turn out to vote. By revealed preferences, any citizen choosing to vote must have a payoff greater than if she abstained (and strictly so unless she is the marginal voter). Adding the payoffs of those who abstain and those who vote, the overall ex-ante expected payoff under majority rule is strictly greater than under random decision making.

Now, let us turn to our problem. In light of the lesson from Börgers (2004), Theorem 2 does not appear obvious any more. If in Börgers's world, holding an election (thereby allowing those who gain from voting to cast their ballot) Pareto-dominates tossing a fair coin, shouldn't a similar argument go through in our case? Shouldn't we let those who gain from voting turn out to vote, instead of tossing a biased coin that picks the bigger party with probability 1?

The problem is, in an asymmetric preference world, the benefit from voting comes at the expected cost from making the "wrong" choice. Mistakes are possible in a Poisson framework because the number of votes for party P is Poisson-distributed with mean $p_n(P)\alpha(P)n$ while the number of abstaining P-supporter is Poisson-distributed with mean $(1 - p_n(P))\alpha(P)n$. Importantly, these two numbers are independently distributed. As n becomes large, the variances of both distributions become large as well. Thus, even conditional on the event that the relative turnout of B to A exceeds a certain threshold, the relative total numbers of A supporters can still be bigger.[15] The benefit from voting applies only to the voters, whose proportion vanishes as n increases. Meanwhile, the cost of a mistake is borne by the whole population. Thus the expected loss from making the "wrong" choice outweighs the benefit from voting for a large population. It is worth noticing that Theorem 2 does not depend on the margin enjoyed by the majority. Whether it is a 51% or a 99% majority, ex-ante, society is better off imposing that choice directly.

Theorem 2 assumes $\alpha(A) > 1/2$. What if $\alpha(A) = 1/2$? In that case, as discussed earlier, the expected benefit from abstaining is independent of the voting rule adopted; consequently, the ex-ante expected social payoff is maximized when the ex-ante expected net marginal benefit from voting is. As the latter maximization will be the object of our analysis in Sect. 4.3, we hold off dealing with the case of $\alpha(A) = 1/2$ till then.

[15] We thank an anonymous referee for suggesting this intuition.

4.3 Turnout and welfare

The analysis of large but finite populations unveils a stark chasm between welfare and turnout, as the former is maximized under a deterministic rule, i.e., by minimizing the latter. This is particularly surprising as the extant literature has instead suggested a coincidence between turnout and welfare maximization for the case of a symmetric electorate (Börgers 2004; Kartal 2015; Faravelli and Sanchez-Pages 2015). To make sense of this apparent inconsistency, consider the ex-ante expected net marginal benefit from voting for a P-supporter, given by the following

$$u_n(P) = p_n(P)U_n(P) - \int_0^{U_n(P)} c\mathrm{d}F(c).$$

As we are going to show, identifying the GMR that maximizes $u_n(P)$ will enable us to unveil the link between turnout and welfare.

Theorem 3 (Voting Benefit Maximization for large n) *Given α, for each $\theta \neq \theta^*$, there exists an N such that $u_n(P)$ for each party P under a θ^*-rule is strictly higher than that under a θ-rule for all $n > N$.*

The intuition behind this result is the following. A θ^*-rule induces the highest pivotal probabilities for both parties, which in turn yield the highest gross marginal benefit from voting. The net marginal benefit is positively related to the gross and thus is maximized at a θ^*-rule.

Theorem 3 is fundamental in two ways. First, it allows us to provide an answer to the welfare maximization problem for the case of $\alpha(A) = 1/2$. Recall that when the electorate is evenly split the expected benefit from abstaining does not depend on the voting rule; consequently, ex-ante expected welfare is maximized by the voting rule maximizing the ex-ante expected net marginal benefit from voting. Hence, as a corollary of Theorem 3, it follows that in a symmetric environment simple majority rule maximizes ex-ante expected social payoff. The discontinuity of the optimal voting rule at $\alpha(A) = 1/2$ stems from the failure of lower hemi-continuity of the optimal social choice as a correspondence of $\alpha(A)$.

Corollary 1 *If $\alpha(A) = 1/2$, for each $\theta \neq 1/2$, there exists an N such that a $1/2$-rule yields a strictly higher ex-ante expected social payoff than a θ-rule for all $n > N$.*

Second, as the expected turnout rate is positively related to the gross marginal benefit from voting, Theorem 3 can be recast in terms of turnout:

Proposition 4 *Given α, for each $\theta \neq \theta^*$, the turnout rate under a θ^*-rule is strictly higher than that under a θ-rule for sufficiently large n.*

In particular, when $\alpha(A) = 1/2$ simple majority rule maximizes turnout. Reading Proposition 4 in light of Corollary 1 and Theorem 2 leads to the observation that the expected payoff is maximized when turnout is maximized *if and only if* the electorate is evenly split. The suggestive concomitance of turnout and welfare (see, Kartal 2015) is thus purely an artefact of symmetry, a coincidence of two features when $\alpha(A) =$

1/2: first, the abstaining benefit is the same for all GMRs; second, $\theta^* = 1/2$ when $\alpha(A) = 1/2$. In general, there is no particular relationship between expected payoff and turnout. Even for the obvious relationship between voting benefit and turnout, the extension from a symmetric electorate under simple majority occurs along the θ^*-rule, rather than other thresholds.

5 Electoral regimes

We conclude our analysis, in the same spirit as Börgers's (2004) seminal paper, with a welfare comparison between voluntary voting and two extreme, and antithetic, electoral regimes: compulsory voting and random decision making.

5.1 Subsidized and compulsory voting

So far we have been assuming that voting is voluntary. But is voluntary voting necessarily a "good" electoral institution? Krasa and Polborn (2009) identify two sources of externalities of voting. When one more citizen casts her ballot, her vote produces a positive externality because (ex-ante) it mitigates the underdog effect and improves the chances of electing the majoritarian option. It also produces a negative externality because an extra vote reduces the pivotal probabilities of any other vote and hence the marginal voting benefits to other voting citizens. This leads to the question of whether subsidizing voting may improve social welfare.

Theorem 4 (Subsidized Voting) *Suppose $\alpha(A) \geq 1/2$. For each $\theta < \theta^*$, for each subsidy amount $s > 0$, voluntary voting under a θ-rule without subsidy yields a strictly higher limiting ex-ante expected social payoff than voting with subsidy s (under the same rule).*

Intuitively, subsidizing voting dampens the underdog effect, making it easier for the majoritarian option to win. However, the majoritarian choice wins with probability 1 at the limit even under voluntary voting whenever $\theta < \theta^*$. The gain from compulsory voting is therefore diminishing as the population grows. The higher social voting cost (or the cost of the subsidy), on the other hand, persists even at the limit. As the voting cost consideration eventually dominates, a subsidy becomes suboptimal. Moreover, as the inequality for the limiting payoffs is strict, the same comparison also holds for all sufficiently large n.

Krasa and Polborn (2009, Proposition 4) evaluate the optimality of subsidized voting against voluntary voting under simple majority rule, drawing a different conclusion than our Theorem 4. This is because in their model the minimum voting cost is bounded away from zero. As such, to give citizens incentives to vote, the "wrong" choice must be chosen with positive probability even at the limit. The positive externality, therefore, prevails in a large population.

Since one can think of compulsory voting as an extreme form of subsidized voting (where the subsidy is large enough to induce all citizens to vote),[16] the following corollary is immediate.

Corollary 2 (Compulsory Voting) *Suppose $\alpha(A) \geq 1/2$. For each $\theta < \theta^*$, voluntary voting under a θ-rule yields a strictly higher limiting ex-ante expected social payoff than compulsory voting under the same rule.*

Compulsory voting has been criticized for its inability to protect the minority or even to reflect the strength of citizens' preferences (see Krishna and Morgan 2015, p. 346). In this light, one may consider the comparison between compulsory and voluntary voting under the *same* electoral threshold as an unfair one. Nevertheless, the next proposition indicates that compulsory voting under an $\alpha(A)$-rule, which protects the minority by imposing a higher threshold, is still welfare inferior to voluntary voting because it is too costly.

Proposition 5 (Compulsory Voting at an α-rule) *Suppose $\alpha(A) \geq 1/2$. Voluntary voting under a θ^*-rule yields a strictly higher limiting ex-ante expected social payoff than compulsory voting under an $\alpha(A)$-rule.*

Note that when $\alpha(A) > 1/2$, any θ-rule with $\theta < \theta^*$ yields a strictly higher expected limiting social payoff than a θ^*-rule under voluntary voting (Theorem 1). Thus Proposition 5 also implies that voluntary voting under any such θ-rule—including simple majority—yields a strictly higher ex-ante expected social payoff than compulsory voting under an $\alpha(A)$-rule.

When the electorate is evenly split (i.e., $\alpha(A) = 1/2$), Proposition 5 implies the following:

Corollary 3 *If $\alpha(A) = 1/2$, voluntary voting under simple majority yields a strictly higher limiting expected social welfare than compulsory voting under the same rule.*

Börgers (2004) shows that, given a symmetric electorate, voluntary voting interim Pareto dominates compulsory voting under a symmetric voting rule. Interestingly, Proposition 5 suggests that Börgers's intuitions can be extended along the comparison of compulsory voting under an α-rule and voluntary voting under a θ^*-rule.

5.2 Random decision regimes

If compulsory voting is too costly, it is natural to wonder whether the other extreme regime, i.e., not holding any election at all, may be welfare superior. The short answer is that an A-deterministic rule—which involves no voting—is optimal in terms of ex-ante expected social welfare, as indicated by Theorem 2.

Nevertheless, implementing an A-deterministic requires knowledge of the identity of A. (Moreover, even if the identity of A is known, an A-deterministic rule may be difficult to advocate.) In light of this, we would like to consider two random social

[16] Voting behavior is driven by the marginal benefit of voting over abstaining. Thus, punishing abstention is the same as subsidizing voting.

decision-making regimes that do not require prior knowledge on the identity of A. The first one is flipping a fair coin. The second one is a random dictatorship—a citizen is randomly picked and her preferred alternative is selected.[17]

Theorem 5 (Random Decision-making) *Suppose* $\alpha(A) > 1/2$. *For each* $\theta < \theta^*$,

1. *Voluntary voting under a* θ*-rule yields a strictly higher limiting ex-ante expected social payoff than flipping a fair coin.*
2. *Voluntary voting under a* θ*-rule yields a strictly higher limiting ex-ante expected social payoff than a random dictatorship.*

The intuition behind these results is the following. Any θ-rule with $\theta < \theta^*$ elects the majoritarian choice with probability 1 at the limit, which is better than a strictly random choice. In addition, a net voting benefit is captured by voluntary voting.

When $\alpha(A) > 1/2, \theta^* > 1/2$. Theorem 5 then implies that voluntary voting under simple majority — which also requires no information on $\alpha(A)$ for its implementation — is strictly better than flipping a fair coin or a random dictatorship when n is large. This implication provides a justification for the conventional wisdom that an election collects and provides information on society's preferences, even when participation is endogenous.

When $\alpha(A) = 1/2$, there is no advantage from implementing the "correct" choice. Yet, voluntary voting is still superior to random decision making due to the net benefit from voting

Proposition 6 *If* $\alpha(A) = 1/2$, *voluntary voting under any non-dictatorial rule yields a strictly higher ex-ante expected social payoff than random decision-making for all finite* n.

Faravelli et al. (2016) demonstrates that voluntary voting under any symmetric voting rule (i.e., a voting rule that treats the two parties in the same way) interim Pareto dominates random decision-making. Proposition 6 is weaker as it provides only an ex-ante welfare comparison. However, it does extend that result to all non-dictatorial voting rules.

6 Conclusion

We consider the class of generalized majority rules in a general costly voting framework. When the prior support of the two alternatives is known, a continuum of GMRs are optimal for a limiting population. However, the conclusions are different as the population approaches the limit, where a deterministic rule imposing the majority choice is optimal. For the case of large populations, we identified the GMR maximizing turnout for a given preference distribution, thus unveiling the exact link between welfare and turnout which had so far remained unexplained in the literature on costly voting (Börgers 2004; Kartal 2015). Welfare and turnout maximization coincide only in a

[17] Probabilistically, a random dictatorship is equivalent to flipping an unfair coin that allows A to win with probability $\alpha(A)$. Note, however, that a random dictatorship can be implemented without knowing the value of $\alpha(A)$.

symmetric environment, where simple majority rule is optimal, their coincidence being an artefact of symmetry. Finally, we showed that voluntary voting is Pareto-superior to both compulsory voting and random decision making, generalizing Börgers's (2004) contribution.

Throughout this paper, we have maintained that the population is a Poisson random variable. This assumption may perhaps be considered specific insomuch as our results rely on various properties of the Poisson distributions, including Environmental Equivalence (Myerson 1998, Theorem 2) and the Concordance Condition (Krishna and Morgan 2015, Condition 1). Nevertheless, Krishna and Morgan (2015) provide a result [first proven by Roos (1999)] allowing the approximation of pivotal probabilities of any population distribution as a mixture of Poisson distributions. Consequently, we feel confident that the Poisson population model is an adequate benchmark for the analysis undertaken here.

As a concluding remark, we would like to stress that the optimality of a deterministic rule for large but finite population should *never* be interpreted as a justification for authoritarian regimes over democracy. First of all, our result assumes known α, while in most authoritarian regimes, the identity of the utilitarian option is subject to debate. Second, democracy serves many social purposes beyond the implementation of the utilitarian options, which are not captured in our analysis. Thus, while our result may suggest the inferiority of some form of democracy (e.g., direct democracy), it should not be read as a disapproval of the principles of democracy.

A notations and preliminaries

Consider a citizen. Let $v = (v(A), v(B))$ be a profile of votes by all other citizens (where $v(P)$ is the number of votes cast for party P). Define $v(T) = v(A) + v(B)$. The space of all other citizens' vote profiles is $\mathbb{Z}^2_+$. A subset of this space is called an event.

With these notations and given a θ-rule, the three types of pivotal events are:

Type I: Changing from a tie to a win by party A, i.e., $v(A) = \theta v(T)$;
Type II: Changing from a win by party B to a tie, i.e., $v(A) + 1 = \theta\big(v(T) + 1\big)$; and
Type III: Changing from a win by party B to a win by party A, i.e., $v(A) < \theta v(T)$ but $v(A) + 1 > \theta\big(v(T) + 1\big)$.

Possible pivots by an B-vote can be similarly defined.

Given a rational number $\theta \in (0, 1)$, define $z^\theta(A)$, $z^\theta(B)$ and $z^\theta(T)$ to be positive integers such that

$$\theta = \frac{z^\theta(A)}{z^\theta(T)} \qquad \text{where } z^\theta(A) \text{ and } z^\theta(T) \text{ are co-prime; and}$$
$$z^\theta(B) = z^\theta(T) - z^\theta(A).$$

When $\theta = 0$, define $z^\theta(A) = 0$ and $z^\theta(B) = 1$. Similarly, when $\theta = 1$, define $z^\theta(A) = 1$ and $z^\theta(B) = 0$. In both cases, define $z^\theta(T) = z^\theta(A) + z^\theta(B)$. If $\theta > 1/2$, $z^\theta(A) > z^\theta(B)$. The reverse is true if $\theta < 1/2$.

Consider citizen i. The event of an exact tie without i's vote (i.e., a Type I pivot) can be denoted as

$$L_\theta = \left\{ z \in \mathbb{Z}_+^2 : z = \gamma\big(z^\theta(A), z^\theta(B)\big) \text{ for some } \gamma \in \mathbb{Z}_+ \right\}.$$

Let $(x_A, x_B) \in \mathbb{Z}^2$ be a duplet of integers (which may be positive, negative or zero). Define the notation

$$L_\theta - (x_A, x_B) = \left\{ z \in \mathbb{Z}_+^2 : z = (z'_A - x_A, z'_B - x_B) \text{ for some } (z'_A, z'_B) \in L_\theta \right\}.$$

Then the set of all Type II pivotal events to an A-supporter (i.e., her vote changes the election from a B-win to a tie) is $L_\theta - (1, 0)$, while that to a B-supporter is $L_\theta - (0, 1)$. Notice that if $\theta = 0$, $L_\theta - (1, 0) = \emptyset$ since there can never be a B-win. Similarly, $L_\theta - (0, 1) = \emptyset$ when $\theta = 1$.

If $z^\theta(B) \leq 1$, there can be no Type III pivotal event for an A-supporter. If instead $z^\theta(B) > 1$, define, for each $z = 1, \ldots, z^\theta(B) - 1$

$$x(z) = \lfloor z \frac{\theta}{1-\theta} \rfloor$$

(where the notation $\lfloor x \rfloor$ denotes the largest integer smaller than x). Then the set of all Type III pivotal events for an A-supporter is given by (see Krishna and Morgan 2015, Appendix C)

$$\bigcup_{1 \leq z \leq z^\theta(B)-1} \{L_\theta - (-x(z), -z)\}.$$

Similarly, there is no Type III pivotal event for a B-supporter when $z^\theta(A) \leq 1$. If instead $z^\theta(A) > 1$, define, for each $z = 1, \ldots, z^\theta(A) - 1$

$$y(z) = \lfloor z \frac{1-\theta}{\theta} \rfloor.$$

The set of all Type III pivotal events for a B supporter is

$$\bigcup_{1 \leq z \leq z^\theta(A)-1} \{L_\theta - (-z, -y(z))\}.$$

Under a 1/2-rule (e.g., simple majority), $z^\theta(A) = z^\theta(B) = 1$, so a Type III pivotal event is impossible.

Lemma 1 allows us to reinterpret the game as a Poisson voting game *without* abstention with population mean $t_n n$, where $t_n = \sum_P \alpha(P) p_n(P)$. Denote this reinterpreted population mean as $T_n = t_n n$. In this reinterpreted Poisson game party P's share of votes is given by

$$\tau_n(P) = \frac{\alpha(P)p_n(P)}{t_n} \quad \text{for } P = A, B.$$

By definition, $\tau_n(A) + \tau_n(B) = 1$ for all n. Write also $\tau(P) = \lim_n \tau_n(P)$.

We will approximate the pivotal probabilities and voting benefits using the techniques in Myerson (2000). As in Myerson (2000), if f_n and g_n are functions of n, the notation $f_n \approx g_n$ means $\lim_n f_n/g_n = 1$.

Lemma 2 *Suppose $\theta, \rho \in (0, 1)$. The equilibrium probability of L_θ can be approximated by*

$$\Pr[L_\theta] \approx \frac{e^{T_n m_n}}{z^\theta(T)\sqrt{2\pi T_n \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} \theta(1-\theta)}},$$

where

$$m_n = \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} - \big(\tau_n(A) + \tau_n(B)\big).$$

Proof This is a direct application of Myerson (2000). □

Remark 1 For all $\theta \in (0, 1)$, $0 \geq m_n \geq -1$. Also, $m_n = 0$ if and only if

$$\frac{\tau_n(A)}{\tau_n(B)} = \frac{\theta}{1-\theta}.$$

Proof Because $\tau_n(A) + \tau_n(B) = 1$ for all n, we can also write

$$m_n = \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} - 1.$$

Now the first term of m_n is a weighted geometric average of $\frac{\tau_n(A)}{\theta}$ and $\frac{\tau_n(B)}{1-\theta}$ with weights θ and $1-\theta$, respectively. As it is always positive, $m_n \geq -1$.

Meanwhile, the second term is the arithmetic average of the same values with the same weights. Hence $m_n \leq 0$ for all n, with equality only when $\tau_n(A)/\theta = \tau_n(B)/(1-\theta)$. □

The next lemma gives the approximation of expected gross marginal benefits from voting.

Lemma 3 *Suppose $\theta, \rho \in (0, 1)$. Given a (θ, ρ)-rule, define*

$$\beta = \frac{1-\theta}{\theta}\frac{\tau(A)}{\tau(B)}$$
$$r(A) = (1-\rho) + \beta^{-(1-\theta)}\rho + \sum_{z \in Z^{\theta}(B)} \beta^{-(\theta z - (1-\theta)x(z))}$$
$$r(B) = \rho + \beta^{\theta}(1-\rho) + \sum_{z \in Z^{\theta}(A)} \beta^{(1-\theta)z - \theta y(z)}.$$

Then for each party P, the equilibrium expected gross marginal benefits from voting is

$$U_n(P) \approx \Pr[L_\theta]\, r(P).$$

Proof The expected gross marginal benefit of voting to an A-supporter is

$$U_n(A) = \Pr[L_\theta](1-\rho) + \Pr[L_\theta - (1,0)] + \sum_{z=1}^{z^{\theta}(B)-1} \rho \Pr[L_\theta - (-x(z), -z)]$$
$$= \Pr[L_\theta]\left\{(1-\rho) + \frac{\Pr[L_\theta - (1,0)]}{\Pr[L_\theta]}\rho + \sum_{z=1}^{z^{\theta}(B)-1} \frac{\Pr[L_\theta - (-x(z), -z)]}{\Pr[L_\theta]}\right\}.$$

Similarly, the expected gross marginal benefit from voting for a B-supporter is

$$U_n(B) = \Pr[L_\theta]\left\{\rho + \frac{\Pr[L_\theta - (0,1)]}{\Pr[L_\theta]}(1-\rho) + \sum_{z=1}^{z^{\theta}(A)-1} \frac{\Pr[L_\theta - (-z, -y(z))]}{\Pr[L_\theta]}\right\}.$$

Using Myerson's (2000) approximation for ratios of the forms $\Pr[L_\theta - (x_A, x_B)] / \Pr[L_\theta]$, we obtain

$$U_n(A) \approx \Pr[L_\theta]\left\{(1-\rho) + \beta^{-(1-\theta)}\rho + \sum_{z=1}^{z^{\theta}(B)-1} \beta^{-(\theta z - (1-\theta)x(z))}\right\}$$
$$U_n(B) \approx \Pr[L_\theta]\left\{\rho + \beta^{\theta}(1-\rho) + \sum_{z=1}^{z^{\theta}(A)-1} \beta^{(1-\theta)z - \theta y(z)}\right\}.$$

□

B optimal voting rules

We first provide a useful expression for the unconditional expected net marginal benefit from voting (i.e., the benefit of voting over the benefit of abstaining, minus the voting cost).

Fact 1 Conditional on supporting P (but not conditional on voting), the ex-ante expected net marginal benefit from voting can be written as

$$u_n(P) = \int_0^{U_n(P)} F(c)\mathrm{d}c.$$

Proof The ex-ante expected net marginal benefit from voting conditional on supporting P is

$$u_n(P) = p_n(P)U_n(P) - \int_0^{U_n(P)} cf(c)\mathrm{d}c.$$

Using the fact that $p_n = F(U_n(P))$ and performing an integration by-parts on the second term gives the desired expression. □

Recall that $\overline{U}_n(P)$ denotes the expected payoff for a P-supporter who abstains. Since abstainers incur no voting cost and receive no voting benefit, $\overline{U}_n(P)$ equals the probability that P wins.

Proof of Theorem 1 The ex-ante expected social payoff at the limit is given by

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n(P) + u_n(P)\right] \\
&= \lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P)\overline{U}_n(P) \\
&= \lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P)\,\Pr[P \text{ wins}].
\end{aligned}$$

When $\alpha(A) > 1/2$, this expression is maximized when A wins with probability 1 at the limit. Theorem 1 now follows from Proposition 2. □

Proof of Theorem 2 Given a (θ, ρ)-rule and an n, write the ex-ante expected payoff to a representative citizen as

$$\begin{aligned}
W_n(\theta,\rho) &= \sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n(P) + u_n(P)\right] \\
&= \sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n(P) + \int_0^{U_n(P)} F(c)\mathrm{d}c\right],
\end{aligned}$$

where the equality follows from Fact 1.

Meanwhile, turnout is zero under an (0, 1)-rule (an A-deterministic rule) as nobody can be pivotal. Thus

$$W_n(0,1) = \alpha(A) \quad \text{for all } n.$$

Hence, for any $(\theta, \rho) \neq (0, 1)$,

$$\begin{aligned} & W_n(0,1) - W_n(\theta,\rho) \\ &= \alpha(A)\left[1 - \overline{U}_n(A)\right] - \alpha(B)\overline{U}_n(B) - \sum_{P\in\{A,B\}} \alpha(P)\int_0^{U_n(P)} F(c)\mathrm{d}c \qquad (2) \\ &= \overline{U}_n(B)\left[\alpha(A) - \alpha(B)\right] - \sum_{P\in\{A,B\}} \alpha(P)\int_0^{U_n(P)} F(c)\mathrm{d}c \qquad (3) \end{aligned}$$

(the second equality follows from $\overline{U}_n(A) + \overline{U}_n(B) = 1$).

If $\theta \in (0, 1)$, Eq. (3) can be rewritten as

$$\Pr[L_\theta]\left\{\left[\alpha(A) - \alpha(B)\right]\frac{\overline{U}_n(B)}{\Pr[L_\theta]} - \sum_{P\in\{A,B\}} \alpha(P)\frac{\int_0^{U_n(P)} F(c)dc}{\Pr[L_\theta]}\right\}. \qquad (4)$$

By Lemma 3,

$$\lim_n \frac{U_n(P)\,F(U_n(P))}{\Pr[L_\theta]} = r(P)F\big(U_n(P)\big) = 0$$

since $F(U_n(P)) \to 0$ for each P. Meanwhile,

$$\begin{aligned} \lim_n \frac{\overline{U}_n(B)}{\Pr[L_\theta]} &= \lim_n \frac{\Pr[B \text{ wins}]}{\Pr[L_\theta]} \\ &= \lim_n \frac{\Pr[L_\theta](1-\rho) + \Pr[B \text{ wins without ties}]}{\Pr[L_\theta]} \quad \geq (1-\rho) > 0 \end{aligned}$$

due to the requirement that $\rho \in (0, 1)$. (It can be shown that the second term in the above limit is also strictly positive, so that the proof goes through even if $\rho = 1$.) Thus the expression in Eq. (4) becomes strictly positive for n sufficiently large.

The above approximation of the benefits applies only to $\theta \in (0, 1)$. However, if $\theta = 1$, at the limit B wins with probability 1 (Proposition 2) while the net voting benefit is 0, so Eq. (3) must be strictly positive for n sufficiently large. Finally, if $\theta = 0$ but $\rho < 1$, it is easy to show that

$$\overline{U}_n(B) = U_n(A) = e^{-T_n}(1-\rho).$$

In this case Eq. (3) becomes

$$\begin{aligned}&\big[\alpha(A)-\alpha(B)\big]e^{-T_n}(1-\rho)-\alpha(A)e^{-T_n}(1-\rho)F\left(e^{-T_n}(1-\rho)\right)\\&\quad=e^{-T_n}(1-\rho)\left[\big[\alpha(A)-\alpha(B)\big]-\alpha(A)F\left(e^{-T_n}(1-\rho)\right)\right],\end{aligned}$$

which is strictly positive for sufficiently large n as $T_n\to\infty$. □

Proof of Theorem 3 By Fact 1, $u_n(P)$ is increasing in the gross marginal voting benefit, $U_n(P)$. We prove Theorem 3, then, by demonstrating in the next lemma that $U_n(P)$ is higher under a θ^*-rule than under any other rule when n is large. □

Lemma 4 *Let $\alpha\geq 1/2$. Consider a voting rule where $\theta,\rho\in(0,1)$ and $\theta\neq\theta^*$. Take any voting rule (θ^*,ρ^*) where $\rho^*\in(0,1)$. Then*

$$\lim_{n\to\infty}\frac{\Pr[L_\theta]}{\Pr[L_{\theta^*}]}=0.$$

Proof Using Lemma 2, and recall that $T_n=nt_n$,

$$\lim_{n\to\infty}\frac{\Pr[L_\theta]}{\Pr[L_{\theta^*}]}=e^{nt_nm_n}\frac{z^{\theta^*}(T)\sqrt{t_n^*\theta^*(1-\theta^*)}}{z^{\theta}(T)\sqrt{t_n(m_n+1)\theta(1-\theta)}}.$$

Suppose by contradiction that this limit is strictly greater than 0. Since $nt_n\to\infty$ (Lemma 1) and $0>m>-1$ (Remark 1), this requires $t_n^*/t_n\to\infty$. Note that

$$\begin{aligned}\frac{t_n^*}{t_n}&=\frac{\sum_P\alpha(P)F\left(U_n^*(P)\right)}{\sum_P\alpha(P)F\left(U_n(P)\right)}\\&=\frac{F\left(\Pr[L_{\theta^*}]\right)\sum_P\alpha(P)\frac{U_n^*(P)}{\Pr[L_{\theta^*}]}}{F\left(\Pr[L_\theta]\right)\sum_P\alpha(P)\frac{U_n(P)}{\Pr[L_\theta]}}.\end{aligned}$$

By Lemma 3, for any $\theta\in(0,1)$, $U_n(P)$ can be approximated by $\Pr[L_\theta]$ multiplied by a number with a finite limit. Hence by Assumption 1, the summations in the numerator and denominator have positive limits. Meanwhile,

$$\begin{aligned}\frac{F\left(\Pr[L_{\theta^*}]\right)}{F\left(\Pr[L_\theta]\right)}&=\frac{F\left(\Pr[L_\theta]\frac{\Pr[L_{\theta^*}]}{\Pr[L_\theta]}\right)}{F\left(\Pr[L_\theta]\right)}\\&\leq\frac{F\left(\Pr[L_\theta]\left(\frac{\Pr[L_{\theta^*}]}{\Pr[L_\theta]}+1\right)\right)}{F\left(\Pr[L_\theta]\right)}.\end{aligned}$$

As we have assumed by contradiction that $\Pr[L_\theta]/\Pr[L_{\theta^*}]\nrightarrow 0$, $\left(\frac{\Pr[L_{\theta^*}]}{\Pr[L_\theta]}+1\right)$ has a strictly positive, finite limit. By Assumption 1, the above expression must have a finite limit, meaning that $t_n^*/t_n\nrightarrow\infty$. □

For any $\theta \in (0, 1)$, Lemma 3 indicates that $U_n(P)$ is equal to $\Pr[L_\theta]$ times $r(P)$, which is finite. Hence Lemma 4 implies

$$\lim_n \frac{U_n(P)}{U_n^*(P)} = 0 \quad \text{for all } P, \text{ for all } \theta \in (0,1)\backslash\{\theta^*\}.$$

To complete the proof of Theorem 3, we tie up the loose ends of the cases when $\theta = 0$ or 1. If the voting rule is deterministic (i.e., $(\theta, \rho) = (0, 1)$ or $(1, 0)$), the voting benefit is zero for all n and Theorem 3 is trivially true. If $\theta = 0$ but $\rho < 1$, then $U_n(B) = 0$ for all n and $\Pr[L_\theta] = e^{-p_n(A)n}$. One can use a similar argument as in the proof of Lemma 4, noting

$$\begin{aligned}\lim_n \frac{\Pr[L_\theta]}{\Pr[L_{\theta^*}]} &= e^{-p_n(A)n} z^{\theta^*}(T)\sqrt{t_n^* n\theta^*(1-\theta^*)}\\ &= e^{-p_n(A)n} z^{\theta^*}(T)\sqrt{\frac{t_n^*}{\alpha(A)p_n(A)}\alpha(A)p_n(A)n\theta^*(1-\theta^*)}.\end{aligned}$$

Using $t_n = \alpha(A)p_n(A)$ when $\theta = 0$, proceed as in the proof of Lemma 4. The case for $\theta' = 1$ but $\rho' > 0$ is similar. □

Proof of Corollary 1 When $\alpha(A) = 1/2$, any voting rule gives the same expected benefit from abstaining. Thus the ex-ante expected social payoff is maximized when the expected net marginal benefit from voting is. Corollary 1 then follows from Theorem 3 and that $\theta^* = 1/2$ when $\alpha(A) = 1/2$. □

Proof of Proposition 4 Note that the expected aggregate turnout rate is given by

$$t_n = \sum_{P\in\{A,B\}} \alpha(P)F(U_n(P)).$$

The proof of Theorem 3 indicates that, for any $\theta \neq \theta^*$, there exists an N such that, for *both* P, $U_n^*(P) > U_n(P)$ for all $n > N$. Thus $t_n^* > t_n$ for all such n as well. □

C electoral regimes

This appendix collects the proofs of theorems in Sect. 5. For this purpose, in this appendix, let $\overline{U}_n^s(P)$ and $U_n^s(P)$ be the abstaining benefit and the gross marginal benefit of voting, respectively, to a P-supporter when there is a subsidy of $s \geq 0$. The case of $s = 0$ corresponds to no subsidy, while $s = 1$ (the highest possible voting cost) corresponds to compulsory voting.

The next fact ensures that the earlier preliminaries apply even when voting is subsidized.

Fact 2 Given a non-dictatorial voting rule, for any $s > 0$ and for $P = A, B$, $U_n^s(P) \to 0$ as $n \to \infty$.

Proof When $s > 0$, a strictly positive fraction of citizen (from each party) will be voting. Hence t_n goes to a strictly positive limit and therefore $T_n \to \infty$. In addition, the limit of $\tau_n(A)/\tau_n(B)$ must be strictly positive and bounded. Hence β is bounded.

The above also mean that we can apply Lemma 2 to approximate $\Pr[L_\theta]$ even when $s > 0$. Since $m_n \leq 0$ and $T_n \to \infty$, It is immediately that $\Pr[L_\theta] \to 0$. As β is bounded, so is $r(P)$. By Lemma 3, $U_n^s(P) \to 0$ for each P. □

C.1 Subsidized and compulsory voting

Proof of Theorem 4 Given a subsidy level s, a (θ, ρ)-rule and an n, write the ex-ante expected payoff to a representative citizen as

$$\begin{aligned} W_n^s(\theta, \rho) &= \sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n^s(P) + F\left(U_n^s(P) + s\right) U_n^s(P) - \int_0^{U_n^s(P)+s} cf(c)\mathrm{d}c \right] \\ &= \sum_{P \in \{A,B\}} \alpha(P) \Bigg[\overline{U}_n^s(P) + F\left(U_n^s(P) + s\right) [U_n^s(P) + s] \\ &\quad - \int_0^{U_n^s(P)+s} cf(c)\mathrm{d}c - F\left(U_n^s(P) + s\right) s \Bigg]. \end{aligned} \tag{5}$$

There are two cases to be considered separately: $s < 1$ or $s \geq 1$. If $s < 1$, by Fact 2, $\lim_n U_n^s(P) + s < 1$ for both P. Using Fact 1, one can rewrite Eq. (5) as

$$W_n^s(\theta, \rho) = \sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n^s(P) + \int_0^{U_n^s(P)+s} F(c)\mathrm{d}c - F\left(U_n^s(P) + s\right) s \right].$$

Hence given a (θ, ρ)-rule and an $s > 0$,

$$\begin{aligned} &\lim_{n\to\infty} W_n^0(\theta, \rho) - W_n^s(\theta, \rho) \\ &\quad = \lim_{n\to\infty} \sum_{P \in \{A,B\}} \alpha(P) \Bigg\{ \left[\overline{U}_n^0(P) - \overline{U}_n^s(P) \right] \\ &\quad\quad - \int_{U_n^0(P)}^{U_n^s(P)+s} F(c)\mathrm{d}c + F\left(U_n^s(P) + s\right) s \Bigg\}. \end{aligned} \tag{6}$$

When $\alpha(A) \geq 1/2$ and $\theta < \theta^*$, by Theorem 1,

$$\sum \alpha(P) \overline{U}_n^0(P) = \alpha(A),$$

which is the maximum possible abstaining benefit. Thus

$$\lim_{n\to\infty} \sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n^0(P) - \overline{U}_n^s(P) \right] \geq 0.$$

Meanwhile, due to Fact 2, for each P,

$$\lim_n F\left(U_n^s(P)+s\right)s - \int_{U_n^0(P)}^{U_n^s(P)+s} F(c)\mathrm{d}c = F(s)s - \int_0^s F(c)\mathrm{d}c > 0,$$

since $F(s) > F(c)$ for all $c < s$. Hence the limit in Eq. (6) is strictly positive.

If $s \geq 1$, Eq. 5 can be rewritten as

$$\sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n^s(P) + [U_n^s(P)+s] - \int_0^1 cf(c)\mathrm{d}c - s\right]$$
$$= \sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n^s(P) + U_n^s(P) - 1 - \int_0^1 F(c)\mathrm{d}c\right]$$

Thus given a (θ, ρ)-rule and an $s > 0$,

$$\begin{aligned}
&\lim_n W_n^0(\theta,\rho) - W_n^s(\theta,\rho)\\
&\quad = \lim_n \sum_{P\in\{A,B\}} \alpha(P)\left\{\left[\overline{U}_n^0(P) - \overline{U}_n^s(P)\right]\right.\\
&\qquad \left. + \int_0^{U_n^0(P)} F(c)\mathrm{d}c - U_n^s(P) + 1 - \int_0^1 F(c)\mathrm{d}c\right\}. \qquad (7)
\end{aligned}$$

Again, the difference in the expected abstaining payoffs is weakly positive. Meanwhile, due to Fact 2, for each P,

$$\lim_n \int_0^{U_n^0(P)} F(c)\mathrm{d}c - U_n^s(P) + 1 - \int_0^1 F(c)\mathrm{d}c = 1 - \int_0^1 F(c)\mathrm{d}c > 0,$$

since $F(c) < 1$ for all $c < 1$. Hence the expression in Eq. (7) must be strictly positive. □

Proof of Corollary 2 Apply Theorem 4, using $s = 1$ (the highest possible voting cost).

Proof of Proposition 5 Take any $\rho, \rho^* \in (0, 1)$. Abusing notations, write $\alpha = \alpha(A)$ for this proof. Using Eq. (7),

$$\begin{aligned}
&\lim_{n\to\infty} W_n^0(\theta^*,\rho^*) - W_n^1(\alpha,\rho)\\
&\quad = \lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P)\left\{\left[\overline{U}_n^0(P;\theta^*) - \overline{U}_n^1(P;\alpha)\right]\right.\\
&\qquad \left. + \int_0^{U_n^0(P;\theta^*)} F(c)dc - U_n^1(P;\alpha) + 1 - \int_0^1 F(c)\mathrm{d}c\right\}.
\end{aligned}$$

Using a version of Proposition 2, it can be shown that, under compulsory voting with a threshold of α,

$$\lim_{n\to\infty} \Pr[A \text{ wins}] = \frac{1}{2},$$

which is the same as under voluntary voting with a threshold of θ^*. Therefore

$$\lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n^0(P;\theta^*) - \overline{U}_n^1(P;\alpha)\right] = 0.$$

Meanwhile, by the same argument as in the proof of Theorem 4,

$$\lim_n \int_0^{U_n^0(P;\theta^*)} F(c)\mathrm{d}c - U_n^s(P;\alpha) + 1 - \int_0^1 F(c)\mathrm{d}c > 0.$$

Hence $\lim_n W_n^0 - W_n^s > 0$. □

Proof of Corollary 3 Apply Proposition 5, noting that $\theta^*(\alpha) = 1/2$ when $\alpha(A) = 1/2$. □

C.2 Random decision-making

Consider a random decision-making regime electing A with probability $q \in (0, 1)$. (Flipping a fair coin corresponds to $q = 1/2$ while a random dictatorship corresponds to $q = \alpha(A)$.) Denote the ex-ante expected social payoff from this regime as

$$W^q = \alpha(A)q + \alpha(B)(1-q).$$

Consider a non-dictatorial (θ, ρ)-rule. For any n,

$$\begin{aligned} &W_n(\theta,\rho) - W^q \\ &\quad= \alpha(A)\left(\overline{U}_n(A) - q\right) + \alpha(B)\left(\overline{U}_n(B) - (1-q)\right) \\ &\quad+ \sum_{P\in\{A,B\}} \alpha(P) \int_0^{U_n(P)} F(c)\mathrm{d}c. \end{aligned}$$

We will use this expression for the next two proofs.

Proof of Theorem 5 When $\theta < \theta^*$, by Proposition 2,

$$\lim_n \overline{U}_n(A) = 1 \quad \text{and} \quad \lim_n \overline{U}_n(B) = 0.$$

Thus

$$\lim_{n\to\infty} W_n(\theta,\rho) - W^q = \big(\alpha(A) - \alpha(B)\big)(1-q) > 0.$$

□

Proof of Proposition 6 When $\alpha(A) = 1/2$,

$$
\begin{aligned}
W_n(\theta, \rho) - W^q &= \frac{1}{2}\left(\overline{U}_n(A) - q\right) + \frac{1}{2}\left(\overline{U}_n(B) - (1-q)\right) \\
&\quad + \frac{1}{2}\sum_{P\in\{A,B\}} \int_0^{U_n(P)} F(c)\mathrm{d}c \\
&= \frac{1}{2}\sum_{P\in\{A,B\}} \int_0^{U_n(P)} F(c)\mathrm{d}c,
\end{aligned}
$$

which is strictly positive for all finite n. □

References

Austen-Smith, D., Banks, J.S.: Information aggregation, rationality, and the condorcet jury theorem. Am. Polit. Sci. Rev. **90**(1), 34–45 (1996)

Barberà, S., Jackson, M.O.: Choosing how to choose: self-stable majority rules and constitutions. Q. J. Econ. **119**(3), 1011–1048 (2004)

Börgers, T.: Costly voting. Am. Econ. Rev. **94**(1), 57–66 (2004)

Buchanan, J., Tullock, G.: The Calculus of Consent. University of Michigan Press, Ann Arbor (1962)

Caplin, A., Nalebuff, B.: On 64%-majority rule. Econometrica **56**(4), 787–814 (1988)

Dasgupta, P., Maskin, E.: On the robustness of majority rule. Journal of the European Economic Association **5**(6), 949–973 (2010)

Downs, A.: An Economic Theory of Democracy. Harper and Row, New York (1957)

Faravelli, M., Sanchez-Pages, S.: (don't) make my vote count. J. Theor. Polit. **27**(4), 544–569 (2015)

Faravelli, M., Man, P., Nguyen, B.D.: Welfare comparison of electoral systems under power sharing. Soc. Choice Welf. **47**(2), 413–429 (2016)

Feddersen, T., Pesendorfer, W.: Convicting the innocent: the inferiority of unanimous jury verdicts under strategic voting. Am. Polit. Sci. Rev. **92**(1), 23–35 (1998)

Graham, B., Bernhardt, D.: Flexibility vs. protection from an unrepresentative legislative majority. Games Econ. Behav. **93**, 59–88 (2015)

Grüner, H.P., Tröger, T.: Linear voting rules. Econometrica **87**(6), 2037–2077 (2019)

Herrera, H., Morelli, M., Palfrey, T.R.: Turnout and power sharing. Econ. J. **124**(574), F131–F162 (2014)

Kartal, M.: A comparative welfare analysis of electoral systems with endogenous turnout. Econ. J. **125**(587), 1369–1392 (2015)

Koriyama, Y., Szentes, B.: A resurrection of the condorcet jury theorem. Theor. Econ. **4**, 227–252 (2009)

Krasa, S., Polborn, M.K.: Is mandatory voting better than voluntary voting? Games Econ. Behav. **66**(1), 275–291 (2009)

Krishna, V., Morgan, J.: Overcoming ideological bias in elections. J. Polit. Econ. **119**(2), 183–211 (2011)

Krishna, V., Morgan, J.: Voluntary voting: costs and benefits. J. Econ. Theory **147**(6), 2083–2123 (2012)

Krishna, V., Morgan, J.: Majority rule and utilitarian welfare. Am. Econ. J. Microeconom. **7**(4), 339–375 (2015)

Ledyard, J.: The paradox of voting and candidate competition: a general equilibrium analysis. In: Horwich, G., Quirk, J.P. (eds.) Essays in Contemporary Fields of Economics. Purdue University Press, West Lafayette (1981)

Ledyard, J.: The pure theory of two candidate elections. Public Choice **44**(1), 7–41 (1984)

Levine, D.K., Palfrey, T.R.: The paradox of voter participation? A laboratory study. Am. Polit. Sci. Rev. **101**(1), 143–158 (2007)

May, K.O.: A set of independent necessary and sufficient conditions for simple majority decision. Econometrica **20**(4), 680–684 (1952)

Messner, M., Polborn, M.K.: Voting on majority rules. Rev. Econ. Stud. **71**, 115–132 (2004)

Messner, M., Polborn, M.K.: The option to wait in collective decisions and optimal majority rules. J. Public Econ. **96**, 524–540 (2012)
Myerson, R.B.: Population uncertainty and Poisson games. Int. J. Game Theory **27**, 375–392 (1998)
Myerson, R.B.: Large Poisson games. J. Econ. Theory **94**, 7–45 (2000)
Palfrey, T.R., Rosenthal, H.: A strategic calculus of voting. Public Choice **41**(1), 7–53 (1983)
Palfrey, T.R., Rosenthal, H.: Voter participation and strategic uncertainty. Am. Polit. Sci. Rev. **79**(1), 62–78 (1985)
Roos, B.: On the rate of multivariate poisson convergence. J. Multivar. Anal. **69**(1), 120–134 (1999)
Taylor, C., Yildirim, H.: A unified analysis of rational voting with private values and group-specific costs. Games Econ. Behav. **70**(2), 457–471 (2010)