GENERALIZED MAJORITY RULES

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Abstract. Generalized majority rules are electoral rules in which an alternative needs to obtain a fixed percentage (not necessarily 50%) of all votes in order to win. This paper analyzes strategic voting behavior under generalized majority rules in a two-alternative, costly voting model with population uncertainty. We find the optimal generalized majority rules for limiting populations, large but finite populations and for social planners who are agnostic about the prior preference distribution, respectively. Examining the properties of these optimal rules allows us to pinpoint the relationship between voter participation and welfare.

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1. Introduction

Consider a two-alternative election. A generalized majority rule (GMR) is a voting rule consisting of two components. First, an electoral threshold specifying the minimum share of the votes that one of the alternatives must exceed in order to win; second, a tie-breaking rule specifying this alternative's winning probability in case its share of the votes is exactly equal to the threshold. The set of GMRs includes, among others, simple majority rule, supermajority rules[1] and selecting one of the alternatives regardless of the voting outcome, which we call dictatorship. This paper investigates strategic costly voting behavior under GMRs in a private value setting, allowing for any possible prior support for the two alternatives. We show that, for large but finite populations, ex-ante expected utility is maximized by dictatorship of the majority. Limiting expected utility, however, is maximized by a continuum of GMRs going from dictatorship of the majority to a critical threshold that partly handicaps the majority (i.e., lying between simple majority and the ex-ante prior support of the majoritarian alternative). Voter turnout is maximized by a GMR coinciding with such critical threshold. If instead the planner is unaware of the prior supports, simple majority rule is optimal.

Majority rules with thresholds different from one-half are not unusual in two-alternative elections. For example, the Constitution of the United States can be amended either by

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[1]Throughout the paper we will use indifferently the terms supermajority and qualified majority.

a two-thirds vote in each house or a proposal called by two-thirds of the states and subsequently ratified by three-quarters of the states. A proposal made by entities other than the European Commission is passed in the Council of the European Union if it is approved by 67% of the member states, 74% of the voting weights (in the Council) and 62% of the EU population.[2] In corporate finance, a "supermajority amendment" requires significantly more than half of the shareholders to approve a merger or other major actions of a company. However, little analysis has been done on voluntary, strategic voting behavior in a costly voting setting.

Even if one contends that simple majority rule is "the" majority rule to be considered, studying supermajority rules is still useful for two purposes. First, if a fraction of the population is always voting (i.e., if they derive net utility from voting), and the votes from this sub-population are not evenly split, then the election from the point of view of the remaining citizens would essentially be one with an asymmetric threshold.[3] For this purpose, a model that predicts voting behavior under GMRs is called for. Second, the optimality of simple majority rule is evaluated in light of its alternatives, i.e., GMRs. Thus, a game theoretic model comparing welfare under different GMRs could potentially justify the use of simple majority. This paper is set to achieve both purposes.

We lay out a benchmark analysis of voluntary voting behavior when both the prior support for the alternatives and the electoral rule are asymmetric. The literature on costly voting (Palfrey and Rosenthal, 1983, 1985; Ledyard, 1981, 1984; Taylor and Yildirim, 2010; Herrera et al., 2014) often allows for an asymmetric support, but not for asymmetric voting rules (i.e., voting rules that are not neutral to the naming of the alternatives). Moreover, the few papers that focus on welfare comparison across electoral rules (Börgers, 2004; Krasa and Polborn, 2009; Faravelli and Sanchez-Pages, 2015; Kartal, 2015) only allow for limited asymmetry in the support.[4] By allowing for asymmetry in both dimensions, we show that the results from this literature are special cases of more general theorems linking turnout and welfare in a broader context.

As a generalization of costly voting under simple majority, our model is not immune from the "paradox of voting" (Downs, 1957; Palfrey and Rosenthal, 1985). Nonetheless, by analyzing the basic driving forces of voting decisions, we provide a foundation for extending various solutions to the paradox under simple majority[5] to GMRs. Moreover, since turnout converges to zero at different rates under different GMRs, the welfare implications for large but finite expected population size are considerable.[6] Equipped with

[2] Supermajority rules are used for non-constitutional matters as well. For example, the state of New York requires a three-fifths majority to pass most tax increases.

[3] We thank Sourav Bhattacharya for suggesting this interpretation to us.

[4] Börgers (2004) requires exact symmetry in the support. Krasa and Polborn (2009); Faravelli and Sanchez-Pages (2015); Kartal (2015) give conclusive welfare comparisons only for an "almost symmetric" support and "almost unanimous" society.

[5] See, for example, Riker and Ordeshook (1968); Morton (1991); Feddersen and Sandroni (2006); Evren (2012); Myatt (2012); Faravelli et al. (2015).

[6] See Faravelli et al. (2015) for the significance of convergence rates in a similar model.

the analysis of the rate of convergence, we characterize the optimal GMRs in relation to three different problems: ex-ante expected limiting payoff maximization; ex-ante expected limiting payoff maximization for a social planner who is unaware of the preference distribution; ex-ante expected payoff maximization for large but finite populations.

We begin by analyzing the problem of ex-ante payoff maximization for a limiting population. Because of the persistence of the paradox of voting in the limit, a representative citizen's expected payoff coincides with her payoff from abstaining. As a consequence, if the electorate is evenly split, ex-ante expected welfare is independent of the voting rule. However, if prior supports are asymmetric then welfare is maximized by a continuum of GMRs going from dictatorship of the majority to a critical threshold lying between one-half and the ex-ante support of the majority. Such threshold handicaps the majority party, as it requires more than half of the votes to win; but does so only partly, as the threshold is lower than its prior support. All GMRs falling in the range between dictatorship of the majority and that critical threshold ensure that the majority wins with probability 1 in the limit.

In light of this finding we consider the case of a social planner who is agnostic about the preference distribution. Think, for instance, of a constitution designer who wishes to set a voting rule that will have to be valid indefinitely, a rule that ought to be optimal independently of the preference distribution. We show that this designer can maximize ex-ante expected limiting welfare if and only if she adopts simple majority rule. Indeed, our previous result implies that setting a threshold equal to one-half ensures that the majority wins with probability one in the limit. But it also implies that simple majority rule is the only GMR that does so no matter what side is the majority and what its margin is. Notably, this is a "belief-free" solution: a "correct" choice is guaranteed in the limit regardless of the designer's belief about the prior support. While axiomatic foundations for simple majority rule have been provided assuming exogenous participation (e.g., May, 1952; Dasgupta and Maskin, 2010), our result provides a positive justification for the prevalence of simple majority rule in environments where voters' participation is endogenous.

We then turn our attention to the case of a large but finite population. We show that, for large expected populations, welfare is higher under a dictatorship of the majority (when there is one) than under any other GMR. This is not because a dictatorship picks the "correct" choice more than half of the time — recall that many GMRs select the majoritarian choice with probability 1 in the limit. Meanwhile, there is no net benefit from voting[7] in a dictatorship, as equilibrium turnout is zero. Thus, there is a trade-off between making the "correct" choice more often and reaping more net benefit from voting. As the former dominates the welfare comparison, a dictatorship turns out to be the optimal rule for payoff maximization.

[7]We refer by this to the marginal benefit from voting net of the voting cost.

The analysis of large but finite populations reveals a striking dichotomy between welfare and turnout: the former is maximized under a dictatorship, that is when the latter is minimized. Such a stark disjunction is surprising insomuch as the extant literature on costly voting has instead identified a coincidence between welfare and (voluntary) turnout. This correspondence is already perceivable in Börgers' (2004) seminal contribution which brings to light how, even in a private value framework, given an evenly split electorate, voluntary voting yields higher welfare than random decision making, in spite of the cost of voting. Subsequently, the coincidence between turnout and welfare has been suggested by papers studying turnout and welfare across different electoral rules. Both Kartal (2015) and Faravelli and Sanchez-Pages (2015) establish that in evenly split electorates simple majority rule maximizes both welfare and turnout. To make sense of this seeming inconsistency we study the maximization problem of expected ex-ante voting benefits and demonstrate that, given a prior support for a party, voting benefits are maximized by the GMR coinciding with the critical threshold we discussed earlier. As voter participation is positively related to the benefit from voting, we demonstrate that a GMR at the critical threshold also maximizes turnout. This threshold, which generally lies between one-half and the majority's prior support, is equal to one-half if and only if the population is symmetrically split. At the same time, as previously noted, if the two parties are ex-ante symmetric the expected benefit from abstaining is independent of the voting rule adopted. Thus welfare for a large but finite, and evenly split, electorate will be determined solely by the net benefit of voting. It follows that simple majority rule maximizes both turnout and expected payoff if an only if the prior supports are equal, a coincidence which is purely an artifact of symmetry.

Finally, in the same spirit as Börgers' (2004) welfare analysis of costly voting, we compare voluntary voting under (limiting) optimal GMRs with two antithetic regimes: compulsory voting, on the one hand, and random decision making, on the other. We show that welfare is higher under voluntary voting, thus generalizing Börgers' result to our more general framework.

According to a common sentiment, supermajority rules are desirable whenever a society faces epochal decisions which may subvert the status quo and the defining identity of that community, such as independence referenda or the referendum on the UK's exit from the European Union. Among the reasons behind this sentiment, it is often remarked that, when facing such drastic turning points, qualified majorities may correct for the inability of democratic voting rules (i.e., one man, one vote) to account for different strengths of preferences.[8] Our results, like those of Krishna and Morgan (2015), suggest otherwise. In particular, they show that, when supporters of the two alternatives have heterogeneous strengths of preferences, simple majority is the only rule implementing the ex-ante utility maximizing alternative with probability 1 in the limit, *regardless of the*

[8]See, for example, `https://www.project-syndicate.org/commentary/brexit-democratic-failure-for-uk-by-kenneth-rogoff-2016-06`.

strength of preferences. While we do not deal with heterogeneous strengths of preferences explicitly, it is possible to translate Krishna and Morgan's model into our framework and prove their theorem on the optimality of simple majority as an extension of our results.[9] That being said, we also show that, when the prior supports of the parties are known, many other GMRs are also optimal in the limit. GMRs that are optimal in the limit are also not necessarily optimal for large but finite population. Essentially, Krishna and Morgan (2015) focus on a rule's ability to aggregate information in the presence of private values, while we are more concerned about the social welfare generated under the voting rule.

With regard to the selection of the "correct" choice, our work is related to the vast literature on the Condorcet jury theorem, i.e., the notion that majorities are more likely than individuals to select the "correct" alternative in a common value setting with imperfect information. Of the many versions of the Condorcet jury problem, the most relevant for our paper are those that allow strategic voting (Austen-Smith and Banks, 1996; Koriyama and Szentes, 2009; Krishna and Morgan, 2011, 2012). In particular, Feddersen and Pesendorfer (1998) consider majority rules with different thresholds, showing how any non-unanimous rule results in a lower probability of error than unanimity. Unlike this literature, our primary interest is the social welfare generated by the voting process, instead of the ability to select the "correct" choice. Of course, the two notions are related, but they do not necessarily coincide. Social welfare is a more complex concept in that both the cost and benefit from voting and abstention are considered.

As a final remark, we should note that several economics studies have analyzed different properties of supermajorities, such as their ability to implement only Pareto improvements (Buchanan and Tullock, 1962); their stability against Condorcet cycles (Caplin and Nalebuff, 1988); their role in relation to self-stable constitutions (Barberà and Jackson, 2004); their function as commitment devices or options against dynamic inconsistency (Messner and Polborn, 2004, 2012); their role in protecting citizens from unrepresentative legislators (Graham and Bernhardt, 2015), to name a few. All of these papers are concerned with a particular rationale for supermajorities and each one of them adopts a specific setting highlighting the rationale of interest. We, instead, wish to consider the canonical model of strategic voting and generalize the notion of majority rule, thereby unravelling the connection between voting incentives, participation, and welfare.

The remainder of the paper is organized as follows. Section 2 lays out the model, Section 3 presents GMRs' limiting equilibrium properties, Section 4 identifies the optimal GMRs, Section 5 compares voluntary voting with compulsory voting and random decision making, Section 6 concludes.

[9] See Appendix F for details.

2. Model

2.1. **Set Up.** Two parties, A and B, are running in an election. The generic party will be denoted as P. The number of citizens is a Poisson random variable with mean n. Each citizen has, independently, the same ex-ante probability $\alpha(A) \in [1/2, 1)$ of preferring party A to B and probability $\alpha(B) = 1 - \alpha(A)$ of preferring B to A.[10] Citizens choose simultaneously to vote for (the candidate of) party A, party B or to abstain. If citizen i votes (instead of abstaining) she bears a private voting cost c_i. The voting costs of citizens are identically and independently drawn from the support $[0, 1]$ according to the cumulative distribution function F with a density function f, where $f(c) > 0$ for all $c \in (0, 1)$ (but not necessarily at the end points).[11] The following assumption will be maintained throughout this paper.[12]

Assumption 1. For any real number $x > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{F(\varepsilon x)}{F(\varepsilon)} \in (0, \infty) \text{ and is continuous in } x.$$

Each citizen's party preference and voting cost are private information, but their distributions are commonly known.

A GMR is described by two parameters: a threshold θ, which is a rational number in $[0, 1]$,[13] and a tie-breaking randomization probability $\rho \in (0, 1)$.[14]

Suppose n_A and n_B are the numbers of votes received by A and B, a (θ, ρ)-rule governs

$$\Pr[A \text{ wins} \mid n_A, n_B] = \begin{cases} 1 & \text{if } n_A > \theta(n_A + n_B) \\ \rho & \text{if } n_A = \theta(n_A + n_B) \\ 0 & \text{if } n_A < \theta(n_A + n_B) \end{cases} \text{; and}$$

$$\Pr[B \text{ wins} \mid n_A, n_B] = 1 - \Pr[A \text{ wins} \mid n_A, n_B].$$

A $(1/2, 1/2)$-rule is simple majority rule. While we will restrict our attention to rules where $\rho \in (0, 1)$, we will sometimes consider the extreme case of an A-dictatorship (a $(0, 1)$-rule). Similarly, a $(1, 0)$-rule is a dictatorship of B. Some statements in this paper depend on θ but not ρ. In those cases, we may speak of a θ-rule, which refers to a voting rule with threshold θ and an arbitrary $\rho \in (0, 1)$.

[10] We name the parties so that A is always the majority.

[11] We can allow the support of voting cost to be some $C \subseteq \mathbb{R}_+$ with $\inf C = 0$. However, since the highest possible benefit from voting is 1, citizens with voting costs above 1 will never vote.

[12] This assumption is satisfied by any distribution where $f(0) > 0$ and is bounded, the uniform distribution, beta distributions, power function distributions (i.e., $F(c) = c^\gamma$ for $\gamma > 0$) and triangular distributions with the lower bound at 0.

[13] If one insists on having an irrational θ, our analysis can go through with a sequence of rational θ_k's converging to the desired irrational θ. See footnote 26 in the Appendix.

[14] If $\rho = 1$ and $\theta > 1/2$, there is a curious equilibrium in which no A-supporters vote. This is because, if no B-supporters vote, A wins ($\rho = 1$) regardless of whether any A-supporters vote. If at least one B-supporter votes, given that no other A-supporters vote and $\theta > 1/2$, an extra A-vote cannot be pivotal. Similarly for $\rho = 0$ and $\theta < 1/2$.

A citizen gets a payoff of 1 if her preferred party wins and 0 if her preferred party loses. Each voting citizen pays her voting cost.

2.2. **Equilibrium.** The above defines a Poisson game (Myerson, 1998). A pure strategy function for a citizen is a measurable function assigning an action (vote for A, for B, or abstain) for each party preference and voting cost realization combination in $\{A, B\} \times [0, 1]$. As in standard Poisson games, we will only consider the case in which all citizens use the same pure strategy function.

For each party preference and voting cost realization, voting for the less preferred party is always strictly worse than abstaining regardless of the number of votes cast for the two parties.[15] In addition, the net marginal benefit of voting for one's preferred party (i.e., the payoff from voting for one's preferred party over the payoff from abstaining, minus the voting cost) is strictly decreasing in the voting cost. Therefore, given the expected number of citizens n, any optimal strategy can be described by a pair of cutoff voting costs, $c_n(A)$ and $c_n(B)$, such that a citizen who prefers party P votes for P for all voting costs $c < c_n(P)$ and abstains for all $c > c_n(P)$.[16]

For most of this paper, it would be convenient to work with the probabilities of voting associated with the cutoff costs. Given n, if $c_n(P)$ is the cutoff cost for party P, the probability that a P-supporter votes will be denoted as $p_n(P) = F(c_n(P))$. A voting probability profile is given by $p_n = (p_n(A), p_n(B))$.

The expected gross (i.e., without subtracting the voting cost) marginal benefit from voting (over abstaining) is the expected pivotal benefit. With a GMR, there are three ways in which a vote for A can be pivotal:

Type I: Changing from a tie to a win for party A;

Type II: Changing from a win for party B to a tie; and

Type III: Changing from a win for party B to a win for party A.

Possible pivots by a B-vote can be similarly defined. Figure 1 illustrates these three types of pivotal events for $\theta = 5/8$.

A Type III pivot cannot happen under a simple majority rule. Nonetheless, this can be an important type of pivot for other majority rules. As illustrated in Figure 1, the number of possible Type III pivot profiles are different across the parties. When $\theta > 1/2$, there are more Type III pivot profiles for a B vote than for an A vote. The reverse is true if $\theta < 1/2$.

Write $U_n(P)$ (which depends on the voting probabilities) as the expected gross marginal benefit from voting for a P-supporter. An equilibrium of a voting game with n

[15]Except when $c = 0$, which is a zero probability event.

[16]We will not specify a P-aligned citizen's choice when her voting cost is exactly $c_n(P)$ since this is a zero probability event.

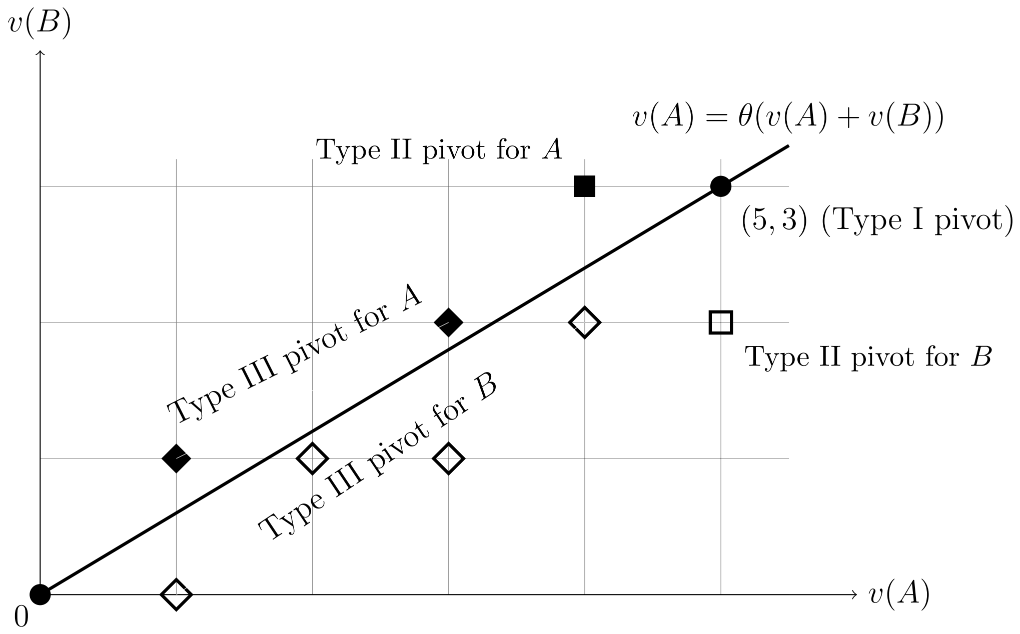

Figure 1. Graphical Illustration of Pivotal Events ($\theta = 5/8$)

expected citizens can be described by a voting profile p_n such that

$$F\left(U_n\left(P\right)\right) = p_n(P) \quad \text{for } P = A, B.$$

For each finite n, an equilibrium of a voting game exists by Theorem 0 of Myerson (2000).

Fixing a (θ, ρ)-rule, let p_n be an equilibrium voting probability profile of a voting game with n expected citizens. Since $p_n \in [0, 1]^2$, we can pick an index set $\mathcal{I}$ for n such that $\{p_n\}_{n\in\mathcal{I}}$ converges. This qualification on the passing to an appropriate index set will be omitted henceforth.

Given a voting probability profile p_n, let $t_n = \sum_{P=A,B} \alpha(P)p_n(P)$ be the expected turnout rate in the voting game with n expected citizens.

3. Equilibrium Analysis

In this section we analyze the limiting equilibrium properties of GMRs. We present four results. Our first object of interest is the number of citizens turning out to vote.

Lemma 1. *For any non-dictatorial (θ, ρ)-rule (i.e., $(\theta, \rho) \neq (0, 1)$ or $(1, 0)$), the expected number of citizens who vote in equilibrium, $t_n n$, goes to infinity as n does.*

While Lemma 1 is a statement about the absolute number of votes cast, our subsequent analysis will be built upon turnout rates, i.e., the proportion of the population that turns out to vote. The next proposition extends the classical paradox of voting, well established for simple majority rule (Palfrey and Rosenthal, 1985), to GMRs.

Proposition 1. *Suppose $\rho \in (0, 1)$. For any (θ, ρ)-rule, the limiting equilibrium turnout rate is zero. That is, $\lim_n p_n(P) = 0$ for each party P.*

As for the case of simple majority rule, turnout rate is zero in the limit because the gross marginal benefit from voting converges to zero as the electorate size increases. Notwithstanding the persistence of the paradox of voting across all generalized majority rules, the proportion of voters for the two parties will decline at different rates as the population increases, thus determining different winning probabilities for the two sides. Our next result provides a characterization of the winning probabilities for a limiting population. This characterization will play a central role in the quest for the optimal GMRs.

Proposition 2. *Given $\alpha(A) \geq 1/2$, there exists a unique $\theta^*(\alpha) \in [1/2, \alpha(A)]$ such that*

$$\lim_{n\to\infty} \Pr\left[A \textit{ wins} \mid n\right] = \begin{cases} 1 & \textit{if } \theta < \theta^*(\alpha) \\ 1/2 & \textit{if } \theta = \theta^*(\alpha) \\ 0 & \textit{if } \theta > \theta^*(\alpha) \end{cases}.$$

In addition, $\theta^*(\alpha) = 1/2$ if only if $\alpha(A) = 1/2$.[17]

It should be noted that the probability of winning at the θ^*-rule is 1/2, rather than ρ, since the randomness is coming from the uncertainty over the population size, preferences and voting cost, rather than from the tie-breaking process. Indeed, the probability of a tie goes to zero as the population grows to infinity, which explains why θ^* and the limiting winning probabilities are independent of ρ.

Two remarks on Proposition 2 are due. First, regardless of the (ex ante) margin enjoyed by the majority, $\theta^* \geq 1/2$ and strictly so in case of a strict majority.[18] Second, θ^* is less than $\alpha(A)$. In other words, for instance, a 2/3-majority will lose with probability 1 when the electoral threshold is 2/3. This property is caused by the "underdog effect", a well known feature of simple majority rule (Levine and Palfrey, 2007) which we extend to GMRs.

Proposition 3 (Underdog Effect)**.** *Suppose $\alpha(A) \geq 1/2$. For any $\theta \in [1/2, \alpha(A)]$, the minority votes more frequently than the majority at a θ-rule. That is,*

$$\lim_{n\to\infty} \frac{p_n(A)}{p_n(B)} \leq 1.$$

At any $\theta \in [1/2, \alpha(A)]$, the underdog effect may be called "partial" (c.f., Herrera et al., 2014) meaning that the majority still receives more than half of the total votes in expectation. However, in terms of *winning probabilities*, the underdog effect is "complete" at the θ^*-rule as the two parties win with equal probabilities, and is "over-turning" when $\theta > \theta^*$, as the minority wins with probability 1.

[17] There is no guarantee that $\theta^*(\alpha)$ is rational, a requirement for an electoral threshold. For practical purposes, however, one can approximate θ^* by a rational number however close one desires. See also footnote 26 on extensions to irrational electoral thresholds.

[18] As it will be clear in our later analysis, this property will be the crux of the argument in the search for a "belief-free" optimal rule (see Corollary 1 in Section 4).

4. Optimal Generalized Majority Rules

Having set out the main properties of generalized majority rules, we are ready to analyze the issue of welfare maximization. In doing so, we consider two different welfare criteria and thus ask two separate questions:

(1) Which voting rule maximizes the ex-ante expected limiting social payoff?
(2) Which voting rule maximizes the ex-ante expected social payoff for large but finite n?

Providing an answer to these questions will allow us to revisit some of the results of the extant literature as special cases of a more general setting, as well as to unveil the actual connection between welfare and turnout which has so far remained obscure in the simple majority framework.

4.1. **Expected Limiting Payoff Maximization.** As we pointed out, Proposition 1 is the consequence of the gross marginal benefit from voting converging to zero, across all GMRs, as the population increases. It follows that a citizen's limiting payoff is given by the payoff from abstaining, which is simply the probability that her preferred option is elected. The representative citizen's limiting ex-ante expected payoff from abstaining (given n) is therefore

$$\lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P) \Pr\left[P \text{ wins} \mid n\right]. \tag{1}$$

We will consider three different cases. First of all, it is immediately clear that when $\alpha(A) = 1/2$ the above expression is equal to $1/2$ independently of what voting rule is applied. Moreover, in this special case, the symmetry between the two parties ensures that the *interim* expected payoff from abstaining coincides with the ex-ante expected payoff from abstaining. Börgers (2004) and Faravelli et al. (2016) get to the same result in a costly voting framework without population uncertainty under symmetric voting rules. Proposition 2 provides a general context against which their results can be interpreted. In particular, it suggests that these previous results are special cases in a richer framework that accommodates asymmetric party support and asymmetric voting rules.

Secondly, let us consider the case of an asymmetrically split electorate. Suppose, with no loss of generality, that $\alpha(A) > 1/2$. In that case, Expression (1) is maximized as long as the probability that A wins goes to 1 as n goes to infinity. The next theorem is an immediate implication of Proposition 2.

Theorem 1 (Expected Limiting Payoff Maximization)**.** *Suppose $\alpha(A) > 1/2$. Any (θ, ρ)-rule with $\theta < \theta^*$ maximizes the expected limiting social payoff.*

Finally, although the above results cover the whole range of possible values of $\alpha(A)$, we ask a different sort of question. Consider the following situation: a social planner wishes to adopt an optimal (or the optimal) electoral rule, but she does not know $\alpha(A)$, which instead is known by the population. For example, a constitution designer may wish to set the voting rule for centuries (at least so hoped) to come. This designer has no idea of which option will be the "majority", lest its percentage support. However, she believes that the electorate will commonly know the preference distribution at the time of the election. Is it possible for this designer to choose a voting rule that maximizes the limiting expected payoff regardless of her beliefs about the preference distribution? The answer is yes and is reported in the following corollary to Theorem 1.

Corollary 1. *A (θ, ρ)-rule maximizes the limiting ex-ante expected social payoff for each and every $\alpha(A) \in (0, 1)$ if and only if $\theta = 1/2$.*[19]

The bite of this corollary lies in the "belief-free" requirement that one rule needs to fit all possible values of α. If A is known to be the majority, but the exact value of α is unknown (i.e., if it were known that $\alpha(A) \geq 1/2$), then any sub-majority rule (i.e., $\theta < 1/2$) would *also* guarantee that A is elected in the limit, thereby maximizing the limiting welfare.

Krishna and Morgan (2015, Theorem 2) show that, when members of the two parties have heterogeneous strengths of preferences, simple majority rule is the only utilitarian supermajority rule (i.e., it always implements the ex-ante payoff maximizing alternative) in the limit. It is possible to translate Krishna and Morgan's model into our framework and prove their Theorem 2 as an extension of Corollary 1. Interested readers may see Appendix F for details.

At first sight, Corollary 1 may appear to evoke the Condorcet jury theorem in that voting under simple majority rule leads to the "correct" decision as the number of voters gets large. Yet, it should be noted that our model is a preference aggregation model (the "correct" choice is commonly known but yields different payoffs to different voters) while the typical Condorcet setting is an information aggregation model (the "correct" choice is imperfectly revealed, but all voters have the same preference). As such, the role of simple majority rule in the two theorems is rather different. In our framework, the optimality of setting the electoral threshold at 1/2 does not lie in its ability to generate an unbiased estimate of the underlying ex-ante preference distribution, as the underdog effect is present even at a 1/2-rule (Proposition 3).[20] Rather, setting $\theta = 1/2$ ensures that the underdog effect cannot overcome the prior advantage of the majority. The role of a majority rule in a Condorcet jury theorem setting depends on the specific version of the theorem (e.g., sincere vs. strategic voting, exogenous vs. endogenous participation) but it typically enables the information of each voter to be aggregated "correctly". This bears little resemblance to the underpinning of Corollary 1.

Corollary 1 offers a theoretical justification for the ubiquity of simple majority rule — it is a "belief-free" solution for a designer who would like to ensure that the majority is

[19] We allow $\alpha(A) \in (0, 1)$ (rather than $[1/2, 1)$ as in the rest of the paper) to reflect the designer's potential ignorance about the identity of the majority party.

[20] Herrera et al. (2014) prove this for the case of a fixed population as well.

elected with probability 1 (in the limit). Unlike traditional axiomatic justifications (e.g., May, 1952; Dasgupta and Maskin, 2010), Corollary 1 acknowledges endogenous voter participation. It confirms that simple majority implements the rationale of aggregative democracy in that it manages to reach a majoritarian decision despite preference-dependent participation.

4.2. **Expected Payoff Maximization for Large n.** As reported in Theorem 1, a continuum of GMRs maximize the limiting expected payoff by implementing the majoritarian choice with probability 1 in the limit. However, expected payoffs under these rules may converge to the limit at different rates. Hence, we ask what voting rule would be optimal for large, but finite n.

Given a (θ, ρ)-rule, let $\overline{U}_n(P)$ be the expected payoff for a P-supporter who abstains. Unlike the limiting case, fixing n, the ex-ante expected payoff to a representative citizen is

$$\sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n(P) + p_n(P) U_n(P) - \int_0^{U_n(P)} c \, dF(c) \right].$$

What voting rule maximizes the above expression for large, but finite n? As we report in the next theorem, the answer to this question is a dictatorship of the majority.

Theorem 2 (Expected Payoff Maximization for large n)**.** *Suppose $\alpha(A) > 1/2$. For each (θ, ρ)-rule, there exists an N such that this rule yields a strictly lower ex-ante expected payoff than an A-dictatorship (i.e., the $(0, 1)$-rule) for all $n > N$.*

The proof of this theorem is not as straight-forward as noting that an A-dictatorship maximizes the limiting ex-ante payoff when $\alpha(A) > 1/2$, since we know that (infinitely) many other GMRs also do so (see Proposition 2). The crux of the argument is instead the following. For a finite n, these other rules guarantee a higher net marginal benefit from voting than a dictatorship, but a lower ex-ante expected benefit from abstaining. As n increases, these two benefits converge to the levels of a dictatorship, but do so at different rates: the former decreases more quickly than the rate at which the latter increases, making a dictatorship of the majority the unique optimal GMR.

It is worth noticing that Theorem 2 does not depend on the margin enjoyed by the majority. Whether it is a 51% or a 99% majority, ex-ante, society is better off imposing that choice directly. Interestingly, this is *not* because of a concern for the cost of voting — there actually is a net benefit from voting arising under any non-dictatorial rule — but because the expected loss from making the "wrong" choice outweighs the benefit from voting.

Theorem 2 assumes $\alpha(A) > 1/2$. What if $\alpha(A) = 1/2$? In that case, as we discussed earlier, the expected benefit from abstaining is independent of the voting rule adopted; consequently, the ex-ante expected social payoff is maximized when the ex-ante expected net marginal benefit from voting is. As the latter maximization will be the object of our analysis in Section 4.3, we hold off dealing with the case of $\alpha(A) = 1/2$ till then.

4.3. **Turnout and Welfare.** The analysis of large but finite populations unveils a stark chasm between welfare and turnout, as the former is maximized under a dictatorship, i.e., by minimizing the latter. This is particularly surprising as the extant literature has instead suggested a coincidence between turnout and welfare maximization for the case of a symmetric electorate (Börgers, 2004; Kartal, 2015; Faravelli and Sanchez-Pages, 2015). To make sense of this apparent inconsistency, consider the ex-ante expected net marginal benefit from voting for a P-supporter, given by the following

$$u_n(P) = p_n(P)U_n(P) - \int_0^{U_n(P)} c\, dF(c).$$

As we are going to show, identifying the GMR that maximizes $u_n(P)$ will enable us to unveil the link between turnout and welfare.

Theorem 3 (Voting Benefit Maximization for large n)**.** *Given α, for each $\theta \neq \theta^*$, there exists an N such that $u_n(P)$ for each party P under a θ^*-rule is strictly higher than that under a θ-rule for all $n > N$.*

The intuition behind this result is the following. A θ^*-rule induces the highest pivotal probabilities for both parties, which in turn yield the highest gross marginal benefit from voting. The net marginal benefit is positively related to the gross and thus is maximized at a θ^*-rule.

Theorem 3 is fundamental in two ways. On the one hand, it allows us to provide an answer to the welfare maximization problem for the case of $\alpha(A) = 1/2$. Recall that when the electorate is evenly split the expected benefit from abstaining does not depend on the voting rule; consequently, ex-ante expected welfare is maximized by the voting rule maximizing the ex-ante expected net marginal benefit from voting. Hence, as a corollary of Theorem 3, it follows that in a symmetric environment simple majority rule maximizes ex-ante expected social payoff. The discontinuity of the optimal voting rule at $\alpha(A) = 1/2$ stems from the failure of lower hemi-continuity of the optimal social choice as a correspondence of $\alpha(A)$.

Corollary 2. *If $\alpha(A) = 1/2$, for each $\theta \neq 1/2$, there exists an N such that a $1/2$-rule yields a strictly higher ex-ante expected social payoff than a θ-rule for all $n > N$.*

On the other hand, as the expected turnout rate is positively related to the gross marginal benefit from voting, Theorem 3 can be recast in terms of turnout:

Proposition 4. *Given α, for each $\theta \neq \theta^*$, the turnout rate under a θ^*-rule is strictly higher than that under a θ-rule for sufficiently large n.*

In particular, when $\alpha(A) = 1/2$ simple majority rule maximizes turnout. Reading Proposition 4 in light of Corollary 2 and Theorem 2 leads to the observation that the expected payoff is maximized when turnout is maximized *if and only if* the electorate is evenly split. The suggestive concomitance of turnout and welfare (Kartal, 2015; Faravelli and Sanchez-Pages, 2015) is thus purely an artifact of symmetry, a coincidence of two features when $\alpha(A) = 1/2$: first, the abstaining benefit is the same for all GMRs; second, $\theta^* = 1/2$ when $\alpha(A) = 1/2$. In general, there is no particular relationship between expected payoff and turnout. Even for the obvious relationship between voting benefit and turnout, the extension from a symmetric electorate under simple majority occurs along the θ^*-rule, rather than other thresholds.

5. Electoral Regimes

We conclude our analysis, in the same spirit as Börgers (2004)' seminal paper, with a welfare comparison between voluntary voting and two extreme, and antithetic, electoral regimes: compulsory voting and random decision making.

5.1. **Subsidized and Compulsory Voting.** So far we have been assuming that voting is voluntary. But is voluntary voting necessarily a "good" electoral institution? Krasa and Polborn (2009) identify two sources of externalities of voting. When one more citizen casts her ballot, her vote produces a positive externality because (ex-ante) it mitigates the underdog effect and improves the chances of electing the majoritarian option. It also produces a negative externality because an extra vote reduces the pivotal probabilities of any other vote and hence the marginal voting benefits to other voting citizens. This leads to the question of whether subsidizing voting may improve social welfare.

Theorem 4 (Subsidized Voting)**.** *Suppose $\alpha(A) \geq 1/2$. For each $\theta < \theta^*$, for each subsidy amount $s > 0$, voluntary voting under a θ-rule without subsidy yields a strictly higher limiting ex-ante expected social payoff than voting with subsidy s (under the same rule).*

Intuitively, subsidizing voting dampens the underdog effect, making it easier for the majoritarian option to win. However, the majoritarian choice wins with probability 1 at the limit even under voluntary voting whenever $\theta < \theta^*$. The gain from compulsory voting is therefore diminishing as the population grows. The higher social voting cost (or the cost of the subsidy), on the other hand, persists even at the limit. As the voting cost consideration eventually dominates, a subsidy becomes suboptimal. Moreover, as the inequality for the limiting payoffs is strict, the same comparison also holds for all sufficiently large n.

Krasa and Polborn (2009, Proposition 4) evaluate the optimality of subsidized voting against voluntary voting under simple majority rule, drawing a different conclusion than our Theorem 4. This is because in their model the minimum voting cost is bounded away from zero. As such, to give citizens incentives to vote, the "wrong" choice must be chosen with positive probability even at the limit. The positive externality, therefore, prevails in a large population.

Since one can think of compulsory voting as an extreme form of subsidized voting (where the subsidy is large enough to induce all citizens to vote),[21] the following corollary is immediate.

Corollary 3 (Compulsory Voting)**.** *Suppose $\alpha(A) \geq 1/2$. For each $\theta < \theta^*$, voluntary voting under a θ-rule yields a strictly higher limiting ex-ante expected social payoff than compulsory voting under the same rule.*

Compulsory voting has been criticized for its inability to protect the minority or even to reflect the strength of citizens' preferences (see Krishna and Morgan, 2015, p. 346). In this light, one may consider the comparison between compulsory and voluntary voting under the *same* electoral threshold as an unfair one. Nevertheless, the next proposition indicates that compulsory voting under an $\alpha(A)$-rule, which protects the minority by imposing a higher threshold, is still welfare inferior to voluntary voting because it is too costly.

Proposition 5 (Compulsory Voting at an α-rule)**.** *Suppose $\alpha(A) \geq 1/2$. Voluntary voting under a θ^*-rule yields a strictly higher limiting ex-ante expected social payoff than compulsory voting under an $\alpha(A)$-rule.*

Note that when $\alpha(A) > 1/2$, any θ-rule with $\theta < \theta^*$ yields a strictly higher expected limiting social payoff than a θ^*-rule under voluntary voting (Theorem 1). Thus Proposition 5 also implies that voluntary voting under any such θ-rule — including simple majority — yields a strictly higher ex-ante expected social payoff than compulsory voting under an $\alpha(A)$-rule.

When the electorate is evenly split (i.e., $\alpha(A) = 1/2$), Proposition 5 implies the following:

Corollary 4. *If $\alpha(A) = 1/2$, voluntary voting under simple majority yields a strictly higher limiting expected social welfare than compulsory voting under the same rule.*

Börgers (2004) and Faravelli et al. (2016) show that, given a symmetric electorate, voluntary voting interim Pareto dominates compulsory voting under a symmetric voting rule. Unlike us, their models do not consider population uncertainty and thus are able to provide such comparison for any population size. Interestingly, Proposition 5 suggests that Börgers' intuitions can be extended along the comparison of compulsory voting under an α-rule and voluntary voting under a θ^*-rule.

[21]Voting behaviour is driven by the marginal benefit of voting over abstaining. Thus, punishing abstention is the same as subsidizing voting.

5.2. **Random Decision Regimes.** If compulsory voting is too costly, it is natural to wonder whether the other extreme regime, i.e., not holding any election at all, may be welfare superior. The short answer is that an A-dictatorship — which involves no voting — is optimal in terms of ex-ante expected social welfare, as indicated by Theorem 2.

Nevertheless, implementing an A-dictatorship requires knowledge of the identity of A. (Moreover, even if the identity of A is known, an A-dictatorship may be difficult to advocate.) In light of this, we would like to consider two random social decision-making regimes that do not require prior knowledge on the identity of A. The first one is flipping a fair coin. The second one is a random dictatorship — a citizen is randomly picked and her preferred alternative is selected.[22]

Theorem 5 (Random Decision-making)**.** *Suppose $\alpha(A) > 1/2$. For each $\theta < \theta^*$,*

(1) *Voluntary voting under a θ-rule yields a strictly higher limiting ex-ante expected social payoff than flipping a fair coin.*
(2) *Voluntary voting under a θ-rule yields a strictly higher limiting ex-ante expected social payoff than a random dictatorship.*

The intuition behind these results is the following. Any θ-rule with $\theta < \theta^*$ elects the majoritarian choice with probability 1 at the limit, which is better than a strictly random choice. In addition, a net voting benefit is captured by voluntary voting.

When $\alpha(A) > 1/2$, $\theta^* > 1/2$. Theorem 5 then implies that voluntary voting under simple majority — which also requires no information on $\alpha(A)$ for its implementation — is strictly better than flipping a fair coin or a random dictatorship when n is large. This implication provides a justification for the conventional wisdom that an election collects and provides information on society's preferences, even when participation endogenous.

When $\alpha(A) = 1/2$, there is no advantage from implementing the "correct" choice. Yet, voluntary voting is still superior to random decision making due to the net benefit from voting

Proposition 6. *If $\alpha(A) = 1/2$, voluntary voting under any non-dictatorial rule yields a strictly higher ex-ante expected social payoff than random decision-making for all finite n.*

Börgers (2004) and Faravelli et al. (2016) demonstrate that voluntary voting under any symmetric voting rule (i.e., a voting rule that treats the two parties in the same way) interim Pareto dominates random decision-making. Proposition 6 is weaker as it provides only an ex-ante welfare comparison. However, it does extend their result to all non-dictatorial voting rules.

[22] Probabilistically, a random dictatorship is equivalent to flipping an unfair coin that allows A to win with probability $\alpha(A)$. Note, however, that a random dictatorship can be implemented without knowing the value of $\alpha(A)$.

6. Conclusion

We introduced the concept of generalized majority rule and presented a general costly voting framework in order to analyze the welfare and turnout properties of a large family of voting rules, allowing for asymmetry in the preference distribution as well in the electoral rule. The set of GMRs includes simple majority, qualified majorities and dictatorship, among others. The breath of our framework enabled us to interpret the findings from the extant literature as special cases of more general theorems, in turn allowing us to better understand the intuition behind these results.

We demonstrated that a continuum of GMRs are optimal for a limiting population. However, if the social planner is unaware of the preference distribution, then simple majority is the only optimal voting rule. This can be interpreted as a special version of Krishna and Morgan's (2015) Theorem 2. Unlike the axiomatic rationalizations of the prevalence of simple majority rule which assume exogenous participation (e.g., May, 1952), these results provide a justification for the ubiquity of such rule in a game theoretic setting with endogenous turnout.

If instead the population is large, but finite (and prior supports are known) dictatorship of the majority is the only optimal GMR. For the case of large populations, we identified the GMR maximizing turnout for a giving preference distribution, thus unveiling the exact link between welfare and turnout which had so far remained unexplained in the literature on costly voting (Börgers, 2004; Faravelli and Sanchez-Pages, 2015; Kartal, 2015). Welfare and turnout maximization coincide only in a symmetric environment, where simple majority rule is optimal, their coincidence being an artifact of symmetry. The general concomitance is rather between turnout and voting benefit maximization.

Finally, we showed that voluntary voting is Pareto-superior to both compulsory voting and random decision making, a result that, loosely speaking, generalizes Börgers' (2004) contribution to a more general setting.

One final remark about the assumption of population uncertainty is due. Throughout this paper, we have maintained that the population is a Poisson random variable. This assumption may perhaps be considered specific insomuch as our results rely on various properties of the Poisson distributions, including Environmental Equivalence (Myerson, 1998, Theorem 2) and the Concordance Condition (Krishna and Morgan, 2015, Condition 1). Nevertheless, Krishna and Morgan (2015) provide a result (first proven by Roos (1999)) allowing the approximation of pivotal probabilities of any population distribution as a mixture of Poisson distributions. Consequently, we feel confident that the Poisson population model is an adequate benchmark for the analysis undertaken here; any generalization in this direction would be beyond the scope of this paper.

Appendix A. Notations and Organization

Table 1: List of Commonly Used Notations

Notation	Limit (as $n \to \infty$)	Meaning and Remarks
Parameters		
n		Expected population size
A, B		Names of the parties, generic element P
$\alpha = \big(\alpha(A), \alpha(B)\big)$		Prior support probabilities profile of the parties, $\alpha(A) + \alpha(B) = 1$
F, f		Cumulative and probability distribution function of private voting cost
θ		Electoral threshold for A to win
ρ		Probability A wins in case of an exact tie at θ
$z^\theta = \big(z^\theta(A), z^\theta(B)\big)$		Profile of co-prime integers such that $\theta = z^\theta(A)/\big(z^\theta(A) + z^\theta(B)\big)$
$Z^\theta(A), Z^\theta(B)$		Index sets
$x(z)$		For each $z \in Z^\theta(B)$, the largest integer in $\{0, \ldots, z^\theta(A)-1\}$ such that $x(z)/\big(x(z)+z\big) < \theta$
$y(z)$		For each $z \in Z^\theta(A)$, the largest integer in $\{0, \ldots, z^\theta(B)-1\}$ such that $z/\big(y(z)+z\big) > \theta$
L_θ		The event of an exact tie (without the decision-maker's vote) at θ
Equilibrium Objects[23,24]		
$p_n = \big(p_n(A), p_n(B)\big)$		Profile of equilibrium voting probabilities when the expected population is n
$t_n = \sum_{P \in \{A,B\}} \alpha(P) p_n(P)$		Total expected turnout rate

[23] All equilibrium objects depend on the voting probabilities. However, this dependence will be suppressed when no confusion may arise.

[24] Equilibrium objects depend on the voting rule under which they are generated. When there can be no confusion regarding the relevant voting rule, dependence on the voting rule will be suppressed. When comparisons across voting rules are made, the notations will don the same decoration as the voting rule. For examples, p'_n will denote the equilibrium voting probabilities under a (θ', ρ')-rule; t^*_n will be the expected turnout rate under a (θ^*, ρ^*)-rule.

Table 1: List of Commonly Used Notations (Continued)

Notation	Limit ($n \to \infty$)	Meaning and Remarks
$T_n = nt_n$		Expected absolute number of turnout
$\tau_n = \big(\tau_n(A), \tau_n(B)\big)$	$\tau = \big(\tau(A), \tau(B)\big)$	Profile of expected vote shares; $\tau_n(A) = \alpha(A)p_n(A)/t_n$ and $\tau_n(A) + \tau_n(B) = 1$ for all n
$\Pr[L_\theta]$		Probability of the event L_θ (given n), dependence on n will be suppressed
m_n	m	Magnitude of the probability of the event L_θ relative to T_n
$\beta_n = \frac{\tau_n(A)}{\tau_n(B)} \frac{1-\theta}{\theta}$	$\beta = \frac{\tau(A)}{\tau(B)} \frac{1-\theta}{\theta}$	The ratio of the expected vote share ratio to the required vote share ratio (for A to win)
$\theta^*(\alpha)$		A critical electoral threshold, dependence on α will be omitted when the reference is clear
$U_n = \big(U_n(A), U_n(B)\big)$		Profile of expected gross (i.e., without subtracting the voting cost) marginal benefit from voting (over abstaining)
$r = \big(r(A), r(B)\big)$		Profile of limiting ratios of $U_n(P)$ to $\Pr[L_\theta]$
$\overline{U}_n = \big(\overline{U}_n(A), \overline{U}_n(B)\big)$		Profile of expected benefit from abstaining
$u_n = \big(u_n(A), u_n(B)\big)$		Profile of unconditional expected net marginal benefits from voting
$W_n(\theta, \rho)$		Ex-ante expected payoff to a representative citizen under a (θ, ρ)-rule

The rest of the appendices are organised as follows: Appendix B contains the technical tools for approximating pivotal probabilities and payoffs. It also contains the proof of Lemma 1 (Section 3). Appendices C, D and E collect all other omitted proofs in Sections 3, 4 and 5, respectively. Appendix F provides the link between our model and Krishna and Morgan' (2015).

Appendix B. Pivotal Probabilities and Payoff Approximations

This appendix has two parts. Appendix B.1 introduces the terminology and notations of describing pivotal events under generalized majority rules. Appendix B.2 provides Poisson approximations of the pivotal probabilities and hence the expected payoffs. Lemma 1 will also be proved there.

B.1. **Pivotal Events.** Consider a citizen. Let $v = (v(A), v(B))$ be a profile of votes by all other citizens (where $v(P)$ is the number of votes cast for party P). Define $v(T) = v(A) + v(B)$. The space of all opponent[25] vote profiles is $\mathbb{Z}_+^2$. A subset of this space is called an event.

With these notations and given a θ-rule, the three types of pivotal events are:

Type I: Changing from a tie to a win by party A, i.e., $v(A) = \theta v(T)$;

Type II: Changing from a win by party B to a tie, i.e., $v(A) + 1 = \theta\big(v(T) + 1\big)$; and

Type III: Changing from a win by party B to a win by party A, i.e., $v(A) < \theta v(T)$ but $v(A) + 1 > \theta\big(v(T) + 1\big)$.

Possible pivots by an B-vote can be similarly defined.

We would like to formally describe these pivotal events. Given a rational number $\theta \in (0, 1)$, define $z^\theta(A)$, $z^\theta(B)$ and $z^\theta(T)$ to be positive integers such that

$$\theta = \frac{z^\theta(A)}{z^\theta(T)} \qquad \text{where } z^\theta(A) \text{ and } z^\theta(T) \text{ are co-prime; and}$$
$$z^\theta(B) = z^\theta(T) - z^\theta(A).$$

When $\theta = 0$, define $z^\theta(A) = 0$ and $z^\theta(B) = 1$. Similarly, when $\theta = 1$, define $z^\theta(A) = 1$ and $z^\theta(B) = 0$. In both cases, define $z^\theta(T) = z^\theta(A) + z^\theta(B)$. Consider citizen i. The event of an exact tie without i's vote can be denoted as

$$L_\theta = \left\{ z \in \mathbb{Z}_+^2 : z = \gamma\big(z^\theta(A), z^\theta(B)\big) \text{ for some } \gamma \in \mathbb{Z}_+ \right\}.$$

By definition, L_θ is also the set of all Type I pivotal events to citizen i (changing from a tie to a win of her preferred party).

Let $(x_A, x_B) \in \mathbb{Z}^2$ be a duplet of integers (which may be positive, negative or zero). Define the notation

$$L_\theta - (x_A, x_B) = \left\{ z \in \mathbb{Z}_+^2 : z = (z'_A - x_A, z'_B - x_B) \text{ for some } (z'_A, z'_B) \in L_\theta \right\}.$$

Then the set of all Type II pivotal events to an A-supporter (i.e., her vote changes the election from a B-win to a tie) is $L_\theta - (1, 0)$, while that to a B-supporter is $L_\theta - (0, 1)$. Notice that if $\theta = 0$, $L_\theta - (1, 0) = \emptyset$ since there can never be a B-win. Similarly, $L_\theta - (0, 1) = \emptyset$ when $\theta = 1$.

[25]Opponent as in "all other players", not "supporters of the other party".

Next consider a Type III pivotal event for an A-supporter (i.e., her vote changes the election from a B-win to an A-win). First note that such an event can occur only when $v(B)$ is not a multiple of $z^\theta(B)$. Also, given $v(B)$, a Type III pivotal event occurs when $v(A)$ is the largest integer such that $v(A)/\big(v(A)+v(B)\big) < \theta$. Hence define an index set

$$Z^\theta(B) = \begin{cases} \big\{1,\dots,z^\theta(B)-1\big\} & \text{if } z^\theta(B) > 1 \\ \emptyset & \text{otherwise} \end{cases}.$$

For each $z \in Z^\theta(B)$, let $x(z)$ be the largest integer in $\big\{0,\dots,z^\theta(A)-1\big\}$ such that

$$\frac{x(z)}{x(z)+z} < \theta.$$

Then the set of all Type III pivotal events to an A-supporter is given by

$$\bigcup_{z\in Z^\theta(B)} \{L_\theta - (-x(z),-z)\}.$$

Similarly, define

$$Z^\theta(A) = \begin{cases} \big\{1,\dots,z^\theta(A)-1\big\} & \text{if } z^\theta(A) > 1 \\ \emptyset & \text{otherwise} \end{cases}.$$

For each $z \in Z^\theta(A)$, let $y(z)$ be the largest integer in $\big\{0,\dots,z^\theta(B)-1\big\}$ such that

$$\frac{z}{y(z)+z} > \theta.$$

Then the set of all Type III pivotal events to a B-supporter is given by

$$\bigcup_{z\in Z^\theta(A)} \{L_\theta - (-z,-y(z))\}.$$

Under a 1/2-rule (e.g., simple majority), $z^\theta(A) = z^\theta(B) = 1$, so a Type III pivotal event is impossible. Also, if $\theta = 0$ or 1, there is no Type III pivotal event since $z^\theta(A), z^\theta(B) \leq 1$. When they exist, the number of Type III pivot profiles is different for the two parties. If $\theta > 1/2$, $z^\theta(A) > z^\theta(B)$. Hence party B has more Type III pivot profiles than party A. The reverse is true if $\theta < 1/2$.

B.2. **Pivotal Probabilities and Payoffs Approximation.** This subsection of Appendix B can be divided into 4 parts: First, the proof of Lemma 1 and Fact 1 allow us to recast our voting game with abstention into one without (but with a different population), making it easier to apply the Poisson approximation techniques in Myerson (2000). Second, we introduce definitions and facts from Myerson (2000), adapted to our framework, in Definitions 1 and 2 as well as Facts 2 and 3. Next, Fact 4 and Lemma 2 guarantee that the required assumptions for Myerson's theorems are satisfied. Finally Lemmas 3 and 4 gives the desired approximations.

First we will prove Lemma 1, which allows us to simplify the problem.

Proof of Lemma 1. Consider a non-dictatorial (θ, ρ)-rule and suppose by contradiction that $\lim_n t_n n = T$ for some finite T. Then

$$\lim_{n\to\infty} \Pr\left[\text{nobody votes}\right] = e^{-T} > 0.$$

If $\theta \in (0,1)$, a single vote will break a tie when nobody else votes. Thus,

$$\begin{aligned}\lim_{n\to\infty} U_n(A) &\geq e^{-T}(1-\rho) \geq 0\\ \lim_{n\to\infty} U_n(B) &\geq e^{-T}\rho \qquad\quad \geq 0,\end{aligned}$$

with at least one inequality being strict. However, this implies $\lim_n t_n = [\alpha(A)p_n(A) + \alpha(B)p_n(B)] > 0$. But then $nt_n \to \infty$.

If $\theta = 0$, since the rule is non-dictatorial, $\rho < 1$. In this case, when nobody votes, a vote from an A-supporter will change the election from a tie (at which A wins with probability ρ) to an outright win of A. Thus

$$\lim_{n\to\infty} U_n(A) \geq e^{-T}(1-\rho) > 0.$$

But this means $p_n(A) > 0$ and hence $nt_n \to \infty$.

The case for $\theta = 1$ is similar, using the pivotal benefit for a B-supporter (noting that $\rho > 0$ for the rule to be non-dictatorial). □

Lemma 1 allows us to reinterpret the game as a Poisson voting game *without* abstention with population mean $t_n n$, where $t_n = \sum_P \alpha(P)p_n(P)$. Denote this reinterpreted population mean as $T_n = t_n n$. In this reinterpreted Poisson game party P's share of votes is given by

$$\tau_n(P) = \frac{\alpha(P)p_n(P)}{t_n} \quad \text{for } P = A, B.$$

By definition, $\tau_n(A) + \tau_n(B) = 1$ for all n. Write also $\tau(P) = \lim_n \tau_n(P)$.

Lemma 1 says that the total number of votes goes to infinity as the expected population size grows. The next fact guarantees that the number of votes for each party must go to a strictly positive limit, provided that the voting rule is interior.

Fact 1. Suppose $\theta, \rho \in (0,1)$. For each party P, the limiting expected number of votes in equilibrium is strictly positive. That is, $\lim_n T_n\tau_n(P) > 0$ for each P.

Proof. Suppose not and say that $\lim_n T_n\tau_n(A) = 0$. Then the probability that no A-supporter votes goes to 1. For n (or T_n) large, then, the pivotal probability that a B-supporter is concentrated around the event $v = (0,0)$. Hence,

$$\begin{aligned}\lim_n U_n(A) &= \Pr\left[v = (0,0)\right](1-\rho) = e^{-T_n}(1-\rho)\\ \lim_n U_n(B) &= \Pr\left[v = (0,0)\right]\rho \qquad\quad = e^{-T_n}\rho.\end{aligned}$$

But then

$$\lim_n \frac{\tau_n(A)}{\tau_n(B)} = \lim_n \frac{\alpha(A)F\left(U_n(A)\right)}{\alpha(B)F\left(U_n(B)\right)}$$

$$= \lim_n \frac{\alpha(A)F\left(e^{-T_n}(1-\rho)\right)}{\alpha(B)F\left(e^{-T_n}\rho\right)}$$
$$> 0,$$

where the last inequality follows from Assumption 1. As $\tau_n(A)+\tau_n(B)=1$ for all n, this means $\lim_n \tau_n(A) > 0$, contradicting $\lim_n T_n\tau_n(A) = 0$. □

Definitions 1, 2 and Fact 2 below are from Myerson (2000), who gives these statements for any sequence of events in the space of vote profiles. For our purpose, however, we are only interested in their application to L_θ, the set of Type I pivotal events.[26] As a remark, the voting probabilities in these three statement need not be equilibrium probabilities.

Definition 1 (Major Sequence)**.** Let $\{p_n\}$ be a sequence of (not necessarily equilibrium) voting probability profiles. A sequence of vote profiles $\{v_n\}_n \subset L_\theta$ is a *major sequence in* L_θ if

$$\lim_{n\to\infty} \frac{\ln \Pr\left[v_n \mid p_n\right]}{T_n} = \lim_{n\to\infty} \max_{\hat{v}_n \in L_\theta} \frac{\ln \Pr\left[\hat{v}_n \mid p_n\right]}{T_n}.$$

Fact 2 (Myerson 2000, Theorem 1)**.** Let $\{p_n\}_n$ be a sequence of voting probability profiles with its corresponding T_n and $\tau_n(P)$ for $P = A, B$. Then

$$\lim_{n\to\infty} \frac{\ln \Pr\left[L_\theta \mid p_n\right]}{T_n} = \lim_{n\to\infty} \max_{v_n \in L_\theta} \sum_{P\in\{A,B\}} \tau_n(P)\psi\left(\frac{v_n(P)}{T_n\tau_n(P)}\right),$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$\psi(x) = x\left(1-\ln x\right) - 1 = -\int_1^x -\ln y \; dy \quad \text{if } x > 0$$
$$\psi(0) = \lim_{x\to 0} \psi(x) = -1.$$

(The maximization is defined with the convention that

$$\tau_n(P)\psi\left(\frac{v_n(P)}{T_n\tau_n(P)}\right) = 0 \qquad \text{if } T_n\tau_n(P) = 0 \text{ and } v_n(P) = 0$$
$$\tau_n(P)\psi\left(\frac{v_n(P)}{T_n\tau_n(P)}\right) = -\infty \quad \text{if } T_n\tau_n(P) = 0 \text{ and } v_n(P) > 0$$

for $P = A, B$.)

Since $(0,0)$ can be chosen for the maximization problem at every n, Fact 2 immediately implies

$$\lim_{n\to\infty} \frac{\ln \Pr\left[L_\theta \mid p_n\right]}{T_n} \geq \psi(0) = -1.$$

Hence we omit the original requirement that the above limit must be finite in the next definition.

[26] As Myerson's technique applies to sequence of events, if one wishes to accommodate an irrational θ, we can apply the results here to a sequence of events$\{L_{\theta_k}\}_k$ where $\{\theta_k\}_k$ is a sequence of rational number converging to the desired irrational θ. Since any irrational θ is in the interior of $[0,1]$ and the limiting probabilities is continuous in θ in this range, the order of limits (k and n) does not matter.

Definition 2 (Offset)**.** Let $\{p_n\}_n$ be a sequence of (not necessarily equilibrium) voting probability profiles with its corresponding T_n and $\tau_n(P)$ for $P = A, B$. For each party P, call $\eta(P)$ the *limit of P-offset* in L_θ if, for every major sequence $\{v_n\}_n \subset L_\theta$,

$$\eta(P) = \lim_n \frac{v_n(P)}{T_n \tau_n(P)}.$$

Using these definitions, we established the following fact, which will be used repeatedly later. Again, the voting probabilities in this fact need not be equilibrium voting probabilities.

Fact 3. Suppose $\theta \in (0, 1)$. Let $\{p_n\}_n$ be a sequence of (not necessarily equilibrium) voting probability profiles. Suppose $\tau_n(P) > 0$ for each P and each n, then

$$\lim_{n\to\infty} \frac{\ln \Pr[L_\theta \mid p_n]}{T_n} = \lim_n \left(\frac{\tau_n(A)}{\theta}\right)^{\theta} \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} - (\tau_n(A) + \tau_n(B)).$$

And the limit offsets in L_θ are

$$\eta(A) = \lim_n \left(\frac{\tau_n(B)}{\tau_n(A)} \frac{\theta}{1-\theta}\right)^{1-\theta};$$

$$\eta(B) = \lim_n \left(\frac{\tau_n(A)}{\tau_n(B)} \frac{1-\theta}{\theta}\right)^{\theta}.$$

Proof. We will use Fact 2 for this proof. For each n, consider the maximization problem

$$\max_{\gamma_n \in \mathbb{Z}_+} \sum_{P \in \{A,B\}} \tau_n(P) \psi\left(\frac{\gamma_n z^\theta(P)}{T_n \tau_n(P)}\right).$$

Ignoring the integer constraint, an optimal solution satisfies the first order condition:

$$\sum_{P \in \{A,B\}} \tau_n(P) \frac{z^\theta(P)}{T_n \tau_n(P)} \ln\left(\frac{\gamma_n z^\theta(P)}{T_n \tau_n(P)}\right) = 0.$$

Rearranging gives

$$\frac{\gamma_n}{T_n} = \frac{1}{z^\theta(T)} \left(\frac{\tau_n(A)}{\theta}\right)^{\theta} \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta}. \tag{2}$$

Call this expression κ_n. Strict concavity of the objective function and the uniqueness means that any major sequence $\{v_n\}_n$ in L_θ satisfies

$$\lim_n \frac{v_n(P)}{T_n \tau_n(P)} = \lim_n \frac{\kappa_n z^\theta(P)}{\tau_n(P)}.$$

Define

$$\eta_n(P) = \frac{\kappa_n z^\theta(P)}{\tau_n(P)} = \begin{cases} \left(\frac{\tau_n(B)}{\tau_n(A)} \frac{\theta}{1-\theta}\right)^{1-\theta} & \text{if } P = A; \\ \left(\frac{\tau_n(A)}{\tau_n(B)} \frac{1-\theta}{\theta}\right)^{\theta} & \text{if } P = B. \end{cases} \tag{3}$$

Then $\eta(P) = \lim \eta_n(P)$ is the limit of P-offsets in L_θ.

Meanwhile, the value of the maximization is

$$\sum_{P\in\{A,B\}} \tau_n(P)\psi\left(\frac{\kappa_n z^\theta(P)}{\tau_n(P)}\right) = z^\theta(T)\kappa_n - \big(\tau_n(A)+\tau_n(B)\big)$$
$$= \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} - \big(\tau_n(A)+\tau_n(B)\big).$$

Taking limits and using Fact 2 yields the equation in the statement. □

The next fact looks similar to Theorem 2 of Myerson (2000) but is different in several ways. Unlike Myerson's theorem, the next fact does not require $T_n\tau_n(P) \to \infty$ for any P such that $w(P) \neq 0$. Nor would it run into difficulties when the product of limit offsets is of the form $0 \cdot \infty$ (since we deal with the limiting sequence, not just the limit). However, it applies only to the event L_θ, rather than any sequence of events possessing limit offsets, as Myerson's theorem does.

Fact 4. Suppose $\theta \in (0,1)$. Let $w = (w(A), w(B)) \in \mathbb{Z}^2$ be a vector of integers (each of its component can be positive, negative or zero). Consider a sequence of voting profiles $\{p_n\}_n$ such that $\lim_n T_n\tau_n(P) > 0$ for each P. Then

$$\lim_n \frac{\Pr\left[L_\theta - w \mid p_n\right]}{\Pr\left[L_\theta \mid p_n\right]} = \lim_n \prod_{P\in\{A,B\}} \left[\eta_n(P)\right]^{w(P)}$$

where $\eta_n(P)$ is as defined in Equation (3).

Proof. For notational clarity, we will suppress the dependence of the probabilities on p_n. Note that $\tau_n(A) + \tau_n(B) = 1$ for all n. Hence if $\lim_n(\tau_n(A)/\tau_n(B))$ is a strictly positive real number, $\lim_n \tau_n(P) > 0$ for both $P = A, B$. Thus $T_n\tau_n(P) \to \infty$ for each P. Also, the limit P-offset of L_θ, $\eta(P)$, is a strictly positive real number, which allow us to switch the order of limit and product on the right hand side of the equation. The statement is then a direct application of Theorem 2 in Myerson (2000).

So suppose $\lim_n(\tau_n(A)/\tau_n(B)) = 0$. If $(1-\theta)w(A) = \theta w(B)$, the event $L_\theta - w$ is the same as L_θ. Using the expressions of $\eta(P)$ in Fact 3,

$$\lim_n \prod_{P\in\{A,B\}} \left[\eta(P)\right]^{w(P)} = \lim_n \left(\frac{\theta}{1-\theta}\frac{\tau_n(B)}{\tau_n(A)}\right)^{(1-\theta)w(A)} \left(\frac{1-\theta}{\theta}\frac{\tau_n(A)}{\tau_n(B)}\right)^{\theta w(B)}$$
$$= 1$$
$$= \lim_n \frac{\Pr\left[L_\theta - w\right]}{\Pr\left[L_\theta\right]}.$$

Hence we can consider $(1-\theta)w(A) \neq \theta w(B)$. Pick an $\varepsilon > 0$ sufficiently small such that $\varepsilon < \theta, 1-\theta$ and

$$\varepsilon\left(|w(A)| + |w(B)|\right) < \left|\theta w(A) - (1-\theta)w(B)\right|.$$

For each n, define

$$\overline{\gamma}_n = T_n \kappa_n \left(\frac{\tau_n(B)}{\tau_n(A)} \right)^{\varepsilon},$$

where κ_n is as defined in Equation (2). The next claim establishes that, if we are interested in expressions at the limit, we can ignore, for each n, any $\gamma > \overline{\gamma}_n$.

Claim 1. For each $\gamma \in \mathbb{Z}_+$, there exists an N such that $\gamma \leq \overline{\gamma}_n$ for all $n > N$.

Proof. A bit of algebra gives

$$\overline{\gamma}_n = \left[z^{\theta}(T) \theta^{\theta} (1-\theta)^{1-\theta} \right]^{-1} T_n \tau_n(A) \left(\frac{\tau_n(B)}{\tau_n(A)} \right)^{1-\theta+\varepsilon} \to \infty,$$

since $\lim_n T_n \tau_n(A) > 0$ and $\lim_n (\tau_n(A)/\tau_n(B)) = 0$. □

Next define γ_n^* to be the largest integer such that

$$\gamma_n^* \leq T_n \kappa_n \left(\frac{\tau_n(A)}{\tau_n(B)} \right)^{\varepsilon}.$$

The next claim establishes that, if we are interested in the limit, we can ignore, for each n, any $\gamma < \gamma_n^*$.

Claim 2. Let $\zeta = (\zeta(A), \zeta(B)) \in \mathbb{Z}^2$ be a vector of integers (where each component may be positive, negative or zero). If $\{\gamma_n\}_n$ is a sequence of positive integers such that $\gamma_n < \gamma_n^*$ for all sufficiently large n, then

$$\limsup_{n \to \infty} \frac{\Pr\left[\gamma_n z^{\theta} - \zeta\right]}{\Pr\left[\gamma_n^* z^{\theta} - \zeta\right]} = 0.$$

Proof. First note that

$$\begin{aligned}
\gamma_n^* &> T_n \kappa_n \left(\frac{\tau_n(A)}{\tau_n(B)} \right)^{\varepsilon} - 1 \\
&= \left[z^{\theta}(T) \theta^{\theta} (1-\theta)^{1-\theta} \right]^{-1} T_n \tau_n(A) \left(\frac{\tau_n(B)}{\tau_n(A)} \right)^{1-\theta-\varepsilon} - 1 \\
&\to \infty.
\end{aligned}$$

Hence there exists an N such that $\gamma_n^* z^{\theta}(P) - \zeta(P) \geq 0$ for both $P = 1, 2$ when $n > N$, and the denominator of the required ratio will not be zero for such n.

The statement in the claim is obviously true if $\gamma_n z^{\theta}(P) - \zeta(P) < 0$ for at least one P for all sufficiently large n. So suppose $\gamma_n z^{\theta}(P) - \zeta(P) \geq 0$ for both P for infinitely many n. For each n,

$$\begin{aligned}
\frac{\Pr\left[\gamma_n z^{\theta} - \zeta\right]}{\Pr\left[\gamma_n^* z^{\theta} - \zeta\right]} &= \prod_{P \in \{A,B\}} \frac{\left[\gamma_n^* z^{\theta}(P) - \zeta(P)\right]!}{\left[\gamma_n z^{\theta}(P) - \zeta(P)\right]!} \frac{1}{\left[T_n \tau_n(P)\right]^{(\gamma_n^* - \gamma_n) z^{\theta}(P)}} \\
&\leq \prod_{P \in \{A,B\}} \left[\frac{\gamma_n^* z^{\theta}(P) - \zeta(P)}{T_n \tau_n(P)} \right]^{(\gamma_n^* - \gamma_n) z^{\theta}(P)}
\end{aligned}$$

$$\leq \prod_{P\in\{A,B\}} \left[\kappa_n \left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\varepsilon} \frac{z^\theta(P)}{\tau_n(P)} \big(1-\zeta(P)\underline{k}_n(P)\big)\right]^{(\gamma_n^*-\gamma_n)z^\theta(P)},$$

where

$$\underline{k}_n(P) = \left[T_n \kappa_n \left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\varepsilon} z^\theta(P)\right]^{-1} \leq \left[(\gamma_n^*+1) z^\theta(P)\right]^{-1} \to 0.$$

Using the expression of κ_n, $z^\theta(A)$, $z^\theta(B)$ and $z^\theta(T)$,

$$\frac{\Pr\left[\gamma_n z^\theta - \zeta\right]}{\Pr\left[\gamma_n^* z^\theta - \zeta\right]} \leq \left[\left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\varepsilon z^\theta(T)} \big(1-\zeta(A)\underline{k}_n(A)\big)^{z^\theta(A)} \big(1-\zeta(B)\underline{k}_n(B)\big)^{z^\theta(B)}\right]^{\gamma_n^*-\gamma_n}$$

$$\to 0. \qquad \square$$

By Claims 1 and 2 (taking $\zeta = (0,0)$ and $\zeta = w$),

$$\lim_n \frac{\Pr\left[L_\theta - w\right]}{\Pr\left[L_\theta\right]} = \lim_n \frac{\sum_{\gamma_n^* \leq \gamma \leq \overline{\gamma}_n} \Pr\left[\gamma z^\theta - w\right]}{\sum_{\gamma_n^* \leq \gamma \leq \overline{\gamma}_n} \Pr\left[\gamma z^\theta\right]}, \tag{4}$$

the numerator of which can be written as

$$\sum_{\gamma_n^* \leq \gamma \leq \overline{\gamma}_n} \frac{\Pr\left[\gamma z^\theta - w\right]}{\Pr\left[\gamma z^\theta\right]} \Pr\left[\gamma z^\theta\right]. \tag{5}$$

For any $\gamma \in \mathbb{Z}_+$, using the p.m.f. of a Poisson distribution,

$$\prod_{P\in\{A,B\}} \left[\frac{\gamma z^\theta(P) - w(P)}{T_n \tau_n(P)}\right]^{w(P)} \leq \frac{\Pr\left[\gamma z^\theta - w\right] p_n}{\Pr\left[\gamma z^\theta\right] p_n} \leq \prod_{P\in\{A,B\}} \left[\frac{\gamma z^\theta(P)}{T_n \tau_n(P)}\right]^{w(P)}.$$

Consider the lower bound first. Take a γ between γ_n^* and $\overline{\gamma}_n$. If $w(P) \geq 0$,

$$\begin{aligned}\left[\frac{\gamma z^\theta(P) - w(P)}{T_n \tau_n(P)}\right]^{w(P)} &\geq \left[\eta_n(P)\left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\varepsilon} - \frac{w(P)}{T_n\tau_n(P)}\right]^{w(P)} \\ &= \left[\eta_n(P)\left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\varepsilon} (1 - w(P)\underline{k}_n(P))\right]^{w(P)};\end{aligned}$$

and if $w(P) < 0$,

$$\begin{aligned}\left[\frac{\gamma z^\theta(P) - w(P)}{T_n \tau_n(P)}\right]^{w(P)} &\geq \left[\eta_n(P)\left(\frac{\tau_n(B)}{\tau_n(A)}\right)^{\varepsilon} - \frac{w(P)}{T_n\tau_n(P)}\right]^{w(P)} \\ &= \left[\eta_n(P)\left(\frac{\tau_n(B)}{\tau_n(A)}\right)^{\varepsilon} (1 - \frac{w(P)}{\overline{\gamma}_n z^\theta(P)})\right]^{w(P)}.\end{aligned}$$

So adopt the shorthand that, for any integer m,

$$K_n(m,P) = \min\left\{\big(1 - |m|\,\underline{k}_n(P)\big)^{|m|}, \left(1 + \frac{|m|}{\overline{\gamma}_n z^\theta(P)}\right)^{-|m|}\right\}.$$

(Note that $K_n \to 1$ since $\underline{k}_n(P) \to 0$ and $\overline{\gamma}_n \to \infty$.) Regardless of the sign of $w(P)$,

$$\left[\frac{\gamma z^{\theta}(P) - w(P)}{T_n \tau_n(P)}\right]^{w(P)} \geq [\eta_n(P)]^{w(P)} \left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\varepsilon |w(P)|} K_n(w(P), P).$$

Thus a lower bound of Equation (5) is

$$\sum_{\gamma_n^* \leq \gamma \leq \overline{\gamma}_n} \left(\prod_{P \in \{A,B\}} [\eta_n(P)]^{w(P)} \left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\varepsilon |w(P)|} K_n(w(P), P) \right) \Pr\left[\gamma z^{\theta} \mid p_n\right]$$
$$= \left(\frac{1-\theta}{\theta}\right)^{\theta w(B) - (1-\theta) w(A)} \left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\theta w(B) - (1-\theta) w(A) + \varepsilon \sum_P |w(P)|}$$
$$\times \left(\prod_{P \in \{A,B\}} K_n(w(P), P) \right) \sum_{\gamma_n^* \leq \gamma \leq \overline{\gamma}_n} \Pr\left[\gamma z^{\theta}\right].$$

Next consider the upper bound. For any γ between γ_n^* and $\overline{\gamma}_n$,

$$\left[\frac{\gamma z^{\theta}(P)}{T_n \tau_n(P)}\right]^{w(P)} \leq [\eta_n(P)]^{w(P)} \left(\frac{\tau_n(B)}{\tau_n(A)}\right)^{\varepsilon |w(P)|}.$$

Hence an upper bound of Equation (5) is

$$\sum_{\gamma_n^* \leq \gamma \leq \overline{\gamma}_n} \left(\prod_{P \in \{A,B\}} [\eta_n(P)]^{w(P)} \left(\frac{\tau_n(B)}{\tau_n(A)}\right)^{\varepsilon |w(P)|} \right) \Pr\left[\gamma z^{\theta}\right]$$
$$= \left(\frac{1-\theta}{\theta}\right)^{\theta w(B) - (1-\theta) w(A)} \left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\theta w(B) - (1-\theta) w(A) - \varepsilon \sum_P |w(P)|} \sum_{\gamma_n^* \leq \gamma \leq \overline{\gamma}_n} \Pr\left[\gamma z^{\theta}\right].$$

Using Equation (4), then,

$$\lim_n \left(\frac{1-\theta}{\theta}\right)^{\theta w(B) - (1-\theta) w(A)} \left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\theta w(B) - (1-\theta) w(A) + \varepsilon \sum_P |w(P)|} \left(\prod_{P \in \{A,B\}} K_n(w(P), P) \right)$$
$$\leq \lim_n \frac{\Pr\left[L_\theta - w\right]}{\Pr\left[L_\theta\right]}$$
$$\leq \lim_n \left(\frac{1-\theta}{\theta}\right)^{\theta w(B) - (1-\theta) w(A)} \left(\frac{\tau_n(A)}{\tau_n(B)}\right)^{\theta w(B) - (1-\theta) w(A) - \varepsilon \sum_P |w(P)|}.$$

Recall that $K_n(w(P), P) \to 1$ for each P, and ε is picked such that $\varepsilon \sum_P |w(P)| < |\theta w(B) - (1-\theta) w(A)|$. Hence, if $\theta w(B) - (1-\theta) w(A) > 0$, both bounds go to 0; and if $\theta w(B) - (1-\theta) w(A) < 0$, both bounds go to positive infinity. In either case, the limits of both bounds are the same as

$$\lim_n \left(\frac{\tau_n(A)}{\tau_n(B)} \frac{1-\theta}{\theta}\right)^{\theta w(B) - (1-\theta) w(A)} = \lim_n \prod_{P \in \{A,B\}} [\eta_n(P)]^{w(P)}.$$

Switch the names of the two parties to prove the case for $\tau_n(A)/\tau_n(B) \to \infty$. $\square$

The above facts imply that, for any interior voting rule, the limiting vote share ratio between the two parties is a strictly positive real number.

Lemma 2. *For any $\theta, \rho \in (0,1)$, in any equilibrium under the (θ,ρ)-rule,* $\lim_n \frac{\tau_n(A)}{\tau_n(B)} \in (0,\infty)$.

Proof. Let $\{(\tau_n(A), \tau_n(B)\}_n$ be a sequence of equilibrium vote share profiles. First note that $\tau_n(A) + \tau_n(B) = 1$ for all n. Hence the ratio $\tau_n(A)/\tau_n(B)$ has a well-defined limit on the extended positive real line. Our goal is to show that this limit is strictly positive and finite.

Suppose by contradiction that $\lim_n \frac{\tau_n(A)}{\tau_n(B)} = 0$. The expected gross marginal benefit from voting (i.e., the expected benefit from being pivotal) for a B-supporter is

$$\begin{aligned} &U_n(B) \\ &= \Pr[L_\theta]\,\rho + \Pr[L_\theta - (0,1)]\,(1-\rho) + \sum_{z \in Z^\theta(A)} \Pr[L_\theta - (-z,-y(z))] \\ &= \Pr[L_\theta] \left\{ \rho + \frac{\Pr[L_\theta - (0,1)]}{\Pr[L_\theta]}(1-\rho) + \sum_{z \in Z^\theta(A)} \frac{\Pr[L_\theta - (-z,-y(z))]}{\Pr[L_\theta]} \right\}. \end{aligned}$$

By Fact 1, $T_n\tau_n(P)$ must attain a strictly positive limit for each P, so Fact 4 is applicable. Hence,

$$\begin{aligned} \lim_n \frac{\Pr[L_\theta - (0,1)]}{\Pr[L_\theta]} &= \lim_n \left(\frac{1-\theta}{\theta} \frac{\tau_n(A)}{\tau_n(B)} \right)^{\theta} &= 0 \\ \lim_n \frac{\Pr[L_\theta - (-z,-y(z))]}{\Pr[L_\theta]} &= \lim_n \left(\frac{1-\theta}{\theta} \frac{\tau_n(A)}{\tau_n(B)} \right)^{(1-\theta)z - \theta y(z)} &= 0. \end{aligned}$$

(The last equality follows since $\theta y(z) < (1-\theta)z$ by definition.) In other words, for n large, $U_n(B)$ is approximately $\rho \Pr[L_\theta]\, p_n$, and $\rho > 0$ is bounded. Note also that $U_n(A) \geq (1-\rho)\Pr[L_\theta]$.

But now we arrive at a contradiction, as

$$\begin{aligned} \lim_n \frac{\tau_n(A)}{\tau_n(B)} &= \lim_n \frac{\alpha(A) F\big(U_n(A)\big)}{\alpha(B) F\big(U_n(B)\big)} \\ &\geq \lim_n \frac{\alpha(A) F\big((1-\rho)\Pr[L_\theta]\big)}{\alpha(B) F\big(U_n(B)\big)} \\ &= \lim_n \frac{\alpha(A) \frac{F\left((1-\rho)\Pr[L_\theta]\right)}{F\left(\Pr[L_\theta]\right)}}{\alpha(B) \frac{F\left(U_n(B)\right)}{F\left(\Pr[L_\theta]\right)}}. \end{aligned}$$

This expression must be strictly positive — it would be so if $\lim \Pr[L_\theta] > 0$; and if $\lim \Pr[L_\theta] = 0$, Assumption 1 guarantees that the above expression is strictly positive.

Switching the names of the parties give the proof for the case of $\tau_n(A)/\tau_n(B) \to \infty$. □

Now we can approximate the pivotal probabilities and voting benefits. As in Myerson (2000), if f_n and g_n are functions of n, the notation $f_n \approx g_n$ means $\lim_n f_n/g_n = 1$.

First we give an approximation of the probability of L_θ.

Lemma 3. *Suppose $\theta, \rho \in (0,1)$. The equilibrium probability of L_θ can be approximated by*

$$\Pr[L_\theta] \approx \frac{e^{T_n m_n}}{z^\theta(T)\sqrt{2\pi T_n \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} \theta(1-\theta)}},$$

where

$$m_n = \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} - \big(\tau_n(A) + \tau_n(B)\big).$$

Proof. The expression of m_n is given by Fact 3.

Lemma 2 implies that $\tau(P) > 0$ for each party P. By Myerson (2000, Theorem 3), the probability of L_θ can be approximated by

$$\begin{aligned}\Pr[L_\theta] &\approx \frac{e^{T_n m_n}}{\sqrt{2\pi T_n \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} z^\theta(A) z^\theta(B)}} \\ &= \frac{e^{T_n m_n}}{z^\theta(T)\sqrt{2\pi T_n \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} \theta(1-\theta)}}.\end{aligned}$$

□

Remark 1. For all $\theta \in (0,1)$, $0 \geq m_n \geq -1$. Also, $m_n = 0$ if and only if

$$\frac{\tau_n(A)}{\tau_n(B)} = \frac{\theta}{1-\theta}.$$

Proof. By Fact 2, m_n is the value to the maximization

$$\max_{v_n \in L_\theta} \sum_{P \in \{A,B\}} \tau_n(P)\psi\left(\frac{v_n(P)}{T_n \tau_n(P)}\right).$$

Since $(0,0) \in L_\theta$ always, m_n must be greater than $\psi(0) = -1$.

Now the first term of m_n is a weighted geometric average of $\frac{\tau_n(A)}{\theta}$ and $\frac{\tau_n(B)}{1-\theta}$ with weights θ and $1-\theta$, respectively. Meanwhile, the second term is the arithmetic average of the same values with the same weights. Hence $m_n \leq 0$ for all n, with equality only when $\tau_n(A)/\theta = \tau_n(B)/(1-\theta)$. □

Remark 2. Because $\tau_n(A) + \tau_n(B) = 1$ for all n, we can also write

$$m_n = \left(\frac{\tau_n(A)}{\theta}\right)^\theta \left(\frac{\tau_n(B)}{1-\theta}\right)^{1-\theta} - 1.$$

Lemma 4. *Suppose $\theta, \rho \in (0,1)$. Given a (θ, ρ)-rule, define*

$$\beta = \frac{1-\theta}{\theta}\frac{\tau(A)}{\tau(B)}$$

$$r(A) = (1-\rho) + \beta^{-(1-\theta)}\rho + \sum_{z \in Z^\theta(B)} \beta^{-(\theta z - (1-\theta)x(z))}$$

$$r(B) = \rho + \beta^{\theta}(1-\rho) + \sum_{z \in Z^\theta(A)} \beta^{(1-\theta)z - \theta y(z)}.$$

Then for each party P, the equilibrium expected gross marginal benefits from voting is

$$U_n(P) \approx \Pr[L_\theta]\, r(P).$$

Proof. The expected gross marginal benefit of voting to an A-supporter is

$$\begin{aligned} U_n(A) &= \Pr[L_\theta](1-\rho) + \Pr[L_\theta - (1,0)]\,\rho \Pr[L_\theta - (-x(z), -z)] \\ &= \Pr[L_\theta] \left\{ (1-\rho) + \frac{\Pr[L_\theta - (1,0)]}{\Pr[L_\theta]}\rho + \sum_{z \in Z^\theta(B)} \frac{\Pr[L_\theta - (-x(z), -z)]}{\Pr[L_\theta]} \right\}. \end{aligned}$$

Similarly, the expected gross marginal benefit from voting for a B-supporter is

$$U_n(B) = \Pr[L_\theta] \left\{ \rho + \frac{\Pr[L_\theta - (0,1)]}{\Pr[L_\theta]}(1-\rho) + \sum_{z \in Z^\theta(A)} \frac{\Pr[L_\theta - (-z, -y(z))]}{\Pr[L_\theta]} \right\}.$$

Write $\beta_n = \frac{1-\theta}{\theta}\frac{\tau_n(A)}{\tau_n(B)}$ and let $\beta = \lim_n \beta_n$. Using Equation (3),

$$\eta_n(A) = [\beta_n]^{-(1-\theta)}; \qquad \eta_n(B) = [\beta_n]^{\theta}.$$

The ratio of the pivotal probabilities can now be obtained by Fact 4. The equilibrium expected gross marginal benefits from voting can then be approximated by

$$U_n(A) \approx \Pr[L_\theta] \left\{ (1-\rho) + \beta^{-(1-\theta)}\rho + \sum_{z \in Z^\theta(B)} \beta^{-(\theta z - (1-\theta)x(z))} \right\}$$

$$U_n(B) \approx \Pr[L_\theta] \left\{ \rho + \beta^{\theta}(1-\rho) + \sum_{z \in Z^\theta(A)} \beta^{(1-\theta)z - \theta y(z)} \right\}. \qquad \square$$

Remark 3. For any $\theta, \rho \in (0,1)$,
If $\beta \geq 1$,

$$\beta^{-(1-\theta)} z^\theta(B) \leq r(A) \leq z^\theta(B)$$
$$z^\theta(A) \leq r(B) \leq \beta^\theta z^\theta(A);$$

and if $\beta \leq 1$,

$$z^\theta(B) \leq r(A) \leq \beta^{-(1-\theta)} z^\theta(B)$$
$$\beta^\theta z^\theta(A) \leq r(B) \leq z^\theta(A).$$

Moreover, all inequalities are strict unless $\beta = 1$.

Proof. First note that, by the definitions of $x(z)$ and $y(z)$,

$$\begin{aligned} &0 \le \theta z - (1-\theta)x(z) < 1-\theta \quad \text{for all } z \in Z^\theta(B) \\ &0 \le (1-\theta)z - \theta y(z) < \theta \qquad \text{for all } z \in Z^\theta(A). \end{aligned}$$

Hence when $\beta \ge 1$,

$$\begin{aligned} r(A) &\ge \beta^{-(1-\theta)}(1-\rho) + \beta^{-(1-\theta)}\rho + \beta^{-(1-\theta)}\big(z^\theta(B)-1\big) &&= \beta^{-(1-\theta)}z^\theta(B) \\ r(A) &\le (1-\rho) + \rho + \big(z^\theta(B)-1\big) &&= z^\theta(B) \\ r(B) &\ge \rho + (1-\rho) + \big(z^\theta(A)-1\big) &&= z^\theta(A) \\ r(B) &\le \beta^\theta\rho + \beta^\theta(1-\rho) + \beta^\theta\big(z^\theta(A)-1\big) &&= \beta^\theta z^\theta(A). \end{aligned}$$

The case of $\beta \le 1$ is similar and therefore omitted. □

Appendix C. Proofs for Equilibrium Analysis

This appendix contains three parts. First we prove Proposition 1 — that the limiting turnout is zero. Next we prove Proposition 2 in two steps: Lemmas 5 and 6 link the party's winning probabilities to β in the first step. Then Lemma 7 establishes the relationship between β and the electoral threshold. Finally we establish the existence of the underdog effect by proving Proposition 3.

C.1. **Proof of Proposition 1.**

First of all, if (θ, ρ) is a dictatorship (i.e., $(\theta,\rho) = (0,1)$ or $(1,0)$), no voter can never be pivotal and turnout is zero for all n. The limiting turnout must therefore be zero.

If $\theta = 0$ but $\rho \in (0,1)$, a B-supporter can never be pivotal. Hence $p_n(B) = 0$ for all n. An A-supporter is pivotal only when nobody else votes. Her expected gross marginal benefits from voting is therefore

$$U_n(A) = \Pr[v = (0,0)]\,(1-\rho) = e^{-T_n}(1-\rho).$$

By Lemma 1, $T_n \to \infty$ so $U_n(A) \to 0$. Thus $p_n(A) = F(U_n(A))$ goes to zero. The case for $\theta = 1$ and $\rho \in (0,1)$ is similar.

Now suppose $\theta, \rho \in (0,1)$. Due to Lemma 2, β is a strictly positive real number. Thus by Lemma 4, for each P, $U_n(P)$ is of the same order as $\Pr[L_\theta]$. Recall that the magnitude of $\Pr[L_\theta]$ is given by m_n, which is weakly negative by Remark 1. In addition, $T_n \to \infty$ (Lemma 1). By Lemma 3, then, $\Pr[L_\theta] \to 0$, meaning that $p_n(P) \to 0$ for each P.

C.2. **Proof of Proposition 2.**

First we give approximations of the winning probabilities under different values of β.

Lemma 5. *Given a (θ, ρ)-rule where $\theta \in (0,1)$,*
If $\beta > 1$,

$$\Pr[B \text{ wins}] \approx \Pr[L_\theta] \left\{ (1-\rho) + \sum_{r\geq 1} \left(\sum_{z \in Z^\theta(A)} \beta^{(1-\theta)z - \theta y(z) - \theta r} + \beta^{-\theta r} \right) \right\};$$

and if $\beta < 1$,

$$\Pr[A \text{ wins}] \approx \Pr[L_\theta] \left\{ \rho + \sum_{r\geq 1} \left(\sum_{z \in Z^\theta(B)} \beta^{(1-\theta)r - (\theta z - (1-\theta)x(z))} + \beta^{(1-\theta)r} \right) \right\}.$$

Proof. Suppose $\beta > 1$. Define V_B as the event of an "outright win by B", that is,

$$V_B = \left\{ v \in \mathbb{Z}_+^2 : (1-\theta)v(A) < \theta v(B) \right\}.$$

For each $v \in V_B$, either there is a $v^\circ \in L_\theta$ and an integer $r \geq 1$ such that $v = v^\circ - (0, -r)$, or there is a $v^\circ \in L_\theta$, a $z \in Z^\theta(A)$ and a $r \geq 1$ such that $v = v^\circ - (-z, -y(z) - r)$. Hence,

$$\frac{\Pr[V_B]}{\Pr[L_\theta]} = \sum_{r\geq 1} \left(\sum_{z \in Z^\theta(A)} \frac{\Pr[L_\theta - (z, -y(z) - r)]}{\Pr[L_\theta]} + \frac{\Pr[L_\theta - (0, -r)]}{\Pr[L_\theta]} \right).$$

By Fact 4 and Lemma 4,

$$\lim_{n\to\infty} \frac{\Pr[V_B]}{\Pr[L_\theta]} = \sum_{r\geq 1} \left(\sum_{z \in Z^\theta(A)} \beta^{(1-\theta)z - \theta y(z) - \theta r} + \beta^{-\theta r} \right).$$

Since $y(z)$ is defined such that $(1-\theta)z - \theta y(z) < \theta$, all the powers of β in the above expressions are negative. As $\beta > 1$, the above sum is finite.

Hence we can write

$$\begin{aligned} &\Pr[B \text{ wins}] \\ &= \Pr[L_\theta](1-\rho) + \Pr[V_B] \\ &= \Pr[L_\theta] \left\{ (1-\rho) + \sum_{r\geq 1} \left(\sum_{z \in Z^\theta(A)} \beta^{(1-\theta)z - \theta y(z) - r} + \beta^{-\theta r} \right) \right\}. \end{aligned}$$

The case of $\beta < 1$ is similar and is therefore omitted. □

Lemma 5 does not cover the case where $\beta = 1$, which is covered by the next lemma.

Lemma 6. *Given a (θ, ρ)-rule where $\theta \in (0,1)$, if $\beta = 1$, then*

$$\lim_{n\to\infty} \Pr[A \text{ wins}] = \lim_{n\to\infty} \Pr[B \text{ wins}] = \frac{1}{2}.$$

Proof. Given n, let $v_n(P)$ be the number of votes received by party P, which is a random variable. Since $T_n \tau_n(P) \to \infty$ for each P (Fact 1), we can approximate the distribution of $v_n(P)$ by a normal distribution with mean $\mu_n(P) = T_n \tau_n(P)$ and standard deviation $\sqrt{\mu_n(P)}$.

Construct the random variables

$$\begin{aligned}\tilde{v}_n(A) &= (1-\theta)v_n(A) \sim N\left((1-\theta)\mu_n(A), (1-\theta)\sqrt{\mu_n(A)}\right)\\ \tilde{v}_n(B) &= \theta v_n(B) \qquad\quad \sim N\left(\theta\mu_n(B), \theta\sqrt{\mu_n(B)}\right).\end{aligned}$$

Since $\beta = 1$,

$$\lim_n (1-\theta)\mu_n(A) = \lim_n (1-\theta)T_n\tau_n(A) = \lim_n \theta T_n\tau_n(B) = \lim_n \theta\mu_n(B).$$

So for $P = A, B$, let Φ_P and ϕ_P be the c.d.f. and p.d.f. of the limiting distribution of $\tilde{v}_n(P)$ and let $\mu = \lim_n (1-\theta)\mu_n(A) = \lim_n \theta\mu_n(B)$ be the common mean of the two distributions. Now

$$\begin{aligned}\lim_n \Pr[A \text{ wins}] &= \lim_n \Big(\Pr[\tilde{v}_n(A) > \tilde{v}_n(B)] + \Pr[\tilde{v}_n(A) = \tilde{v}_n(B)]\,\rho\Big)\\ &= \int_{-\infty}^{\infty} [1-\Phi_A(v)]\,\phi_B(v)\;dv.\\ \lim_n \Pr[B \text{ wins}] &= \lim_n \Big(\Pr[\tilde{v}_n(A) < \tilde{v}_n(B)] + \Pr[\tilde{v}_n(A) = \tilde{v}_n(B)]\,(1-\rho)\Big)\\ &= \int_{-\infty}^{\infty} \Phi_A(v)\phi_B(v)\;dv.\end{aligned}$$

As both ϕ_A and ϕ_B are symmetric around μ,

$$\begin{aligned}\int_{-\infty}^{\infty} [1-\Phi_A(v)]\,\phi_B(v)\;dv &= \int_{-\infty}^{\infty} [1-\Phi_A(\mu+x)]\,\phi_B(\mu+x)\;dx\\ &= \int_{-\infty}^{\infty} \Phi_A(\mu-x)\phi_B(\mu-x)\;dx\\ &= \int_{-\infty}^{\infty} \Phi_A(v)\phi_B(v)\;dv.\end{aligned}$$

Hence,

$$\lim_n \Pr[A \text{ wins}] = \lim_n \Pr[B \text{ wins}] = \frac{1}{2}. \qquad \square$$

Due to Lemmas 5 and 6, we will first prove the existence of a $\theta^* \in [1/2, \alpha(A)]$ such that $\beta^* = 1$, and $\beta > 1$ whenever $\theta < \theta^*$ and vice versa. Lastly, we will tie up the loose ends and consider the extreme cases of $\theta = 0$ and 1, which are not covered by Lemmas 5 and 6.

We will first prove a useful fact.

Fact 5. Let $\{\varepsilon_n\}$ and $\{\varepsilon'_n\}$ be two sequences of strictly positive numbers converging to zero. Suppose Assumption 1 holds. Then

$$\lim_{n\to\infty} \frac{F\big(\varepsilon_n(1-\theta)\big)}{F\big(\varepsilon_n\theta\big)} \le \lim_{n\to\infty} \frac{F\big(\varepsilon'_n(1-\theta')\big)}{F\big(\varepsilon'_n\theta'\big)}$$

whenever $\theta \ge \theta'$.

Proof. By Assumption 1, for any $\theta \in (0,1)$,

$$\lim_n \frac{F\big(\varepsilon_n(1-\theta)\big)}{F\big(\varepsilon_n\theta\big)} = \lim_n \frac{\frac{F\left(\varepsilon_n(1-\theta)\right)}{F\left(\varepsilon_n\right)}}{\frac{F\left(\varepsilon_n\theta\right)}{F\left(\varepsilon_n\right)}} = \lim_n \frac{\frac{F\left(\varepsilon'_n(1-\theta)\right)}{F\left(\varepsilon'_n\right)}}{\frac{F\left(\varepsilon'_n\theta\right)}{F\left(\varepsilon'_n\right)}} = \lim_{n\to\infty} \frac{F\big(\varepsilon'_n(1-\theta)\big)}{F\big(\varepsilon'_n\theta\big)}. \tag{6}$$

Meanwhile, for any $\varepsilon_n > 0$, the function $\frac{F(\varepsilon_n(1-\theta))}{F(\varepsilon_n\theta)}$ is strictly decreasing in θ. Thus if $\theta \geq \theta'$,

$$\frac{F\big(\varepsilon_n(1-\theta)\big)}{F\big(\varepsilon_n\theta\big)} \leq \frac{F\big(\varepsilon_n(1-\theta')\big)}{F\big(\varepsilon_n\theta'\big)} \qquad \text{for all } n$$

$$\lim_n \frac{F\big(\varepsilon_n(1-\theta)\big)}{F\big(\varepsilon_n\theta\big)} \leq \lim_n \frac{F\big(\varepsilon_n(1-\theta')\big)}{F\big(\varepsilon_n\theta'\big)}. \tag{7}$$

Combining Inequalities (6) and (7) gives the desired statement. ☐

Lemma 7. *Given $\alpha(A) \geq 1/2$, there exists a unique $\theta^*(\alpha) \in [1/2, \alpha(A)]$ such that,*

$$\beta > 1 \quad \textit{whenever } \theta < \theta^*(\alpha)$$
$$\beta = 1 \quad \textit{whenever } \theta = \theta^*(\alpha)$$
$$\beta < 1 \quad \textit{whenever } \theta > \theta^*(\alpha).$$

Moreover, $\theta^* = 1/2$ if and only if $\alpha(A) = 1/2$.

Proof. First we will show that there is a unique θ^* such that $\beta^* = 1$ (regardless of the value of ρ). Note that if $\beta = 1$,

$$\frac{\tau(A)}{\theta} = \frac{\tau(B)}{1-\theta}. \tag{8}$$

Recall from Lemma 3 that the magnitude of $\Pr[L_\theta]$ is given by m_n. By Remark 1, when Equation (8) is satisfied, $\lim m_n = m = 0$. By Remark 2, if $m = 0$, the first term of m must be 1.

Using Lemma 4 and Remark 3,

$$\begin{aligned} U_n(A) &\approx \Pr[L_\theta]\, z^\theta(B) \\ &\approx \frac{z^\theta(B)}{z^\theta(T)\sqrt{2\pi T_n \theta(1-\theta)}} \\ &= \frac{1-\theta}{\sqrt{2\pi T_n \theta(1-\theta)}}. \end{aligned}$$

And similarly,

$$U_n(B) \approx \frac{\theta}{\sqrt{2\pi T_n \theta(1-\theta)}}.$$

Adopt the shorthand $s_n = (2\pi T_n \theta(1-\theta))^{-1/2}$. We look for the solution to

$$\lim_n \frac{\alpha(A) F\left(s_n(1-\theta)\right)}{\alpha(B) F\left(s_n \theta\right)} = \frac{\theta}{1-\theta}. \tag{9}$$

By Fact 5, the left hand side is weakly decreasing in θ and is independent of ρ. Meanwhile, the right hand side is strictly increasing in θ and is independent of ρ. Since $\alpha(A) \geq 1/2$, at $\theta = 1/2$, the left hand side of Equation (9) is greater than 1 (strictly so if $\alpha(A) > 1/2$) while the right hand side is 1 . At $\theta = \alpha(A)$, the left hand side is weakly less than $\alpha(A)/\alpha(B)$ (due to Fact 5) while the right hand side is $\alpha(A)/\alpha(B)$. Since both sides are continuous functions of θ, there exists a unique $\theta^* \in [1/2, \alpha(A)]$ solving Equation (9). In addition, $\theta^* = 1/2$ if and only if $\alpha(A) = 1/2$.

Next we will show that $\beta > 1$ whenever $\theta < \theta^*$. Suppose by contradiction that $\theta < \theta^*$ but $\beta \leq 1$. This means

$$\frac{\tau(A)}{\tau(B)} \leq \frac{\theta}{1-\theta} < \frac{\theta^*}{1-\theta^*}.$$

Using Remark 3 and $\beta \leq 1$,

$$\begin{aligned} r(A) &\geq z^\theta(B) = z^\theta(T)(1-\theta) \\ r(B) &\leq z^\theta(A) = z^\theta(T)\theta. \end{aligned}$$

But by Lemma 4 and since $\theta < \theta^*$

$$\begin{aligned} \frac{\tau(A)}{\tau(B)} &\geq \lim_n \frac{\alpha(A) F\left(\Pr[L_\theta] z^\theta(T)(1-\theta)\right)}{\alpha(B) F\left(\Pr[L_\theta] z^\theta(T)\theta\right)} \\ &\geq \lim_n \frac{\alpha(A) F\left(s_n(1-\theta^*)\right)}{\alpha(B) F\left(s_n \theta^*\right)} && \text{(Fact 5)} \\ &= \frac{\theta^*}{1-\theta^*} && \text{(Equation (9))}. \end{aligned}$$

Contradiction. The proof for the $\theta > \theta^*$ case is similar and is therefore omitted. □

Together with Lemmas 5 and 6, Lemma 7 proves Proposition 2 for all $\theta \in (0,1)$. To complete our proof, we also need to show that, when $\theta = 0$ and 1, A's limiting winning probability is 1 and 0 respectively.

Consider the case where $\theta = 0$. If $\rho = 1$, it is an A-dictatorship and A wins always. If $\rho < 1$, a B-supporter is never pivotal and will never vote. Thus B wins only when nobody votes and the tie is broken in favor of B. This happens with probability

$$\Pr[\text{nobody votes}](1-\rho) = e^{-T_n}(1-\rho).$$

As $T_n \to \infty$ (Lemma 1), this probability goes to 0. The case of $\theta = 1$ is analogous.

C.3. **Proof of Proposition 3.** First suppose $1/2 \leq \theta < \theta^*$. By Lemma 7, $\beta > 1$. Remark 3 implies

$$\lim_n \frac{U_n(A)}{U_n(B)} = \frac{r(A)}{r(B)} < \frac{z^\theta(B)}{z^\theta(A)}.$$

Meanwhile, since $\theta \geq 1/2$, $z^\theta(A) \geq z^\theta(B)$ (see Appendix B.1). Hence the above limit is strictly less than 1. Since the cost distribution is symmetric across parties, $p_n(A) \leq p_n(B)$ for sufficiently large n.

Next consider the case where $\theta^* \leq \theta \leq \alpha$. Note that

$$\lim_n \frac{p_n(A)}{p_n(B)} = \beta \frac{\theta}{1-\theta} \frac{1-\alpha}{\alpha}.$$

Since $\theta \geq \theta^*$, $\beta \leq 1$ (Lemma 7). Meanwhile, as $\theta \leq \alpha$, $\frac{\theta}{1-\theta}\frac{1-\alpha}{\alpha} \leq 1$. Therefore $\lim_n p_n(A)/p_n(B) \leq 1$.

Appendix D. Optimal Voting Rules

This appendix collects the proofs of results in Section 4. We will first give a useful expression for the net marginal voting benefit. Then Appendix D.1 collects the proofs on the limiting payoff results (Theorem 1 and Corollary 1).

Appendix D.2 deals with the results on large but finite n (Theorem 2). Appendix D.3, which collects all the proofs on results on the voting benefits (Theorem 3, Corollary 2 and Proposition 4).

We first provide a useful expression for the unconditional expected net marginal benefit from voting (that is, the benefit of voting over the benefit of abstaining, minus the voting cost).

Fact 6. Conditional on supporting P (but not conditional on voting), the ex-ante expected net marginal benefit from voting can be written as

$$u_n(P) = \int_0^{U_n(P)} F(c)\, dc.$$

Proof. The ex-ante expected net marginal benefit from voting conditional on supporting P is

$$u_n(P) = p_n(P) U_n(P) - \int_0^{U_n(P)} c f(c)\, dc.$$

Using the fact that $p_n = F(U_n(P))$ and performing an integration by-parts on the second term gives the desired expression. □

D.1. **Results on Limiting Payoffs.**

Proof of Theorem 1. The ex-ante expected social payoff at the limit is given by

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n(P) + u_n(P)\right] \\
&= \lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P)\overline{U}_n(P) \\
&= \lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P) \Pr[P \text{ wins}].
\end{aligned}$$

When $\alpha(A) > 1/2$, this expression is maximized when A wins with probability 1 at the limit. Theorem 1 now follows from Proposition 2. □

Proof of Corollary 1. If: If $\alpha(A) = 1/2$, any voting rule maximises the expected limiting social payoff. Meanwhile, by Proposition 2, if $\theta = 1/2$, then $\theta < \theta^*$ if $\alpha(A) > 1/2$ and $\theta < \theta^*$ if $\alpha(A) < 1/2$. In other words, when $\theta = 1/2$, A wins whenever $\alpha(A) > 1/2$ and B wins whenever $\alpha(A) < 1/2$. Hence a 1/2-rule maximizes the limiting ex-ante expected social payoff for each and every $\alpha(A) \in [0, 1]$.
Only if: Due to Proposition 2, any $\theta > 1/2$ fails to select the majority with probability 1 in the limit when $\alpha(A)$ falls between 1/2 and θ. Similarly, any $\theta < 1/2$ fails to select the majority with probability 1 in the limit when $\alpha(A)$ falls between θ and 1/2. Thus $\theta = 1/2$ is the only θ that maximizes the limiting ex-ante expected social payoff for each and every $\alpha(A) \in [0, 1]$. □

D.2. **Result for Large but Finite n.**

Proof of Theorem 2. Given a (θ, ρ)-rule and an n, write the ex-ante expected payoff to a representative citizen as

$$\begin{aligned} W_n(\theta,\rho) &= \sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n(P) + u_n(P)\right] \\ &= \sum_{P\in\{A,B\}} \alpha(P)\left[\overline{U}_n(P) + \int_0^{U_n(P)} F(c)\,dc\right], \end{aligned}$$

where the equality follows from Fact 6.

Meanwhile, turnout is zero under an $(0, 1)$-rule (an A-dictatorship) as nobody can be pivotal. Thus

$$W_n(0,1) = \alpha(A) \quad \text{for all } n.$$

Hence, for any $(\theta, \rho) \neq (0, 1)$,

$$\begin{aligned} &W_n(0,1) - W_n(\theta,\rho) \qquad (10)\\ &= \alpha(A)\left[1 - \overline{U}_n(A)\right] - \alpha(B)\overline{U}_n(B) - \sum_{P\in\{A,B\}} \alpha(P) \int_0^{U_n(P)} F(c)\,dc \\ &= \overline{U}_n(B)\left[\alpha(A) - \alpha(B)\right] - \sum_{P\in\{A,B\}} \alpha(P) \int_0^{U_n(P)} F(c)\,dc \\ &= \overline{U}_n(B)\, p_n^{\theta,\rho}\left[\alpha(A) - \alpha(B)\right] - \sum_{P\in\{A,B\}} \alpha(P)\left[U_n(P)\, F(U_n(P)) + o(U_n(P))\right] \qquad (11) \end{aligned}$$

(the second equality follows from $\overline{U}_n(A) + \overline{U}_n(B) = 1$, while the last equality comes from a first-order Taylor approximation around $\lim_n U_n(P) = 0$).

If $\theta \in (0,1)$, Equation (11) can be rewritten as

$$\Pr[L_\theta]\left\{\big[\alpha(A)-\alpha(B)\big]\frac{\overline{U}_n(B)}{\Pr[L_\theta]}-\sum_{P\in\{A,B\}}\alpha(P)\frac{\int_0^{U_n(P)}F(c)dc}{\Pr[L_\theta]}\right\}. \tag{12}$$

By Lemma 4,

$$\begin{aligned}&\lim_n \frac{U_n(P)\,F(U_n(P))}{\Pr[L_\theta]}\\ &= r(P)F\big(U_n(P)\big)\\ &= 0\end{aligned}$$

since $F(U_n(P)) \to 0$ for each P.

Thus if $\overline{U}_n(B) \not\to 0$, the limit of Equation (12) will be strictly positive. If $\overline{U}_n(B) \to 0$, by Proposition 2 we must have $\theta < \theta^*$. Lemma 7 then indicates $\beta > 1$, in which case Lemma 5 implies

$$\frac{\overline{U}_n(B)}{\Pr[L_\theta]} > 0,$$

meaning that the expression in Equation (12) becomes strictly positive for n sufficiently large. In either case, a dictatorship yields a higher ex-ante expected social payoff than a (θ,ρ)-rule for n sufficiently large.

The above approximation of the benefits applies only to $\theta \in (0,1)$. However, if $\theta = 1$, at the limit B wins with probability 1 (Proposition 2) while the net voting benefit is 0, so Equation (11) must be strictly positive for n sufficiently large. Finally, if $\theta = 0$ but $\rho < 1$, it is easy to show that

$$\overline{U}_n(B) = U_n(A) = e^{-T_n}(1-\rho).$$

In this case Equation (11) becomes

$$\begin{aligned}&\big[\alpha(A)-\alpha(B)\big]e^{-T_n}(1-\rho)-\alpha(A)e^{-T_n}(1-\rho)F\left(e^{-T_n}(1-\rho)\right)\\ &\qquad = e^{-T_n}(1-\rho)\left[\big[\alpha(A)-\alpha(B)\big]-\alpha(A)F\left(e^{-T_n}(1-\rho)\right)\right],\end{aligned}$$

which is strictly positive for sufficiently large n as $T_n \to \infty$. □

D.3. **Results on Voting Benefits.**

Proof of Theorem 3. By Fact 6, $u_n(P)$ is increasing in the gross marginal voting benefit, $U_n(P)$. We prove Theorem 3, then, by demonstrating that $U_n(P)$ is higher under a θ^*-rule than under any other rule when n is large.

Lemma 8. *Consider $\alpha \geq 1/2$. Let (θ,ρ) be a voting rule where $\theta,\rho \in (0,1)$ and $\theta \neq \theta^*$. Also consider a voting rule (θ^*,ρ^*) where $\rho^* \in (0,1)$. Then*

$$\lim_{n\to\infty}\frac{\Pr[L_\theta]}{\Pr[L_{\theta^*}]} = 0.$$

Proof. Using Lemma 3, and recall that $T_n = nt_n$,

$$\lim_{n\to\infty} \frac{\Pr[L_\theta]}{\Pr[L_{\theta^*}]} = e^{nt_n m_n} \frac{z^{\theta^*}(T)\sqrt{t_n^*\theta^*(1-\theta^*)}}{z^{\theta}(T)\sqrt{t_n(m_n+1)\theta(1-\theta)}}.$$

Suppose by contradiction that this limit is strictly greater than 0. Since $nt_n \to \infty$ (Lemma 1) and $0 > m > -1$ (Remark 1), this requires $t_n^*/t_n \to \infty$. Note that

$$\begin{aligned}\frac{t_n^*}{t_n} &= \frac{\sum_P \alpha(P) F\left(U_n^*(P)\right)}{\sum_P \alpha(P) F\left(U_n(P)\right)} \\ &= \frac{F\left(\Pr[L_{\theta^*}]\right)\sum_P \alpha(P)\frac{U_n^*(P)}{\Pr[L_{\theta^*}]}}{F\left(\Pr[L_\theta]\right)\sum_P \alpha(P)\frac{U_n(P)}{\Pr[L_\theta]}}.\end{aligned}$$

By Lemma 4, for any $\theta \in (0,1)$, $U_n(P)$ can be approximated by $\Pr[L_\theta]$ multiplied by a number with a finite limit. Hence by Assumption 1, the summations in the numerator and denominator have positive limits. Meanwhile,

$$\begin{aligned}\frac{F\left(\Pr[L_{\theta^*}]\right)}{F\left(\Pr[L_\theta]\right)} &= \frac{F\left(\Pr[L_\theta]\frac{\Pr[L_{\theta^*}]}{\Pr[L_\theta]}\right)}{F\left(\Pr[L_\theta]\right)} \\ &\le \frac{F\left(\Pr[L_\theta]\left(\frac{\Pr[L_{\theta^*}]}{\Pr[L_\theta]}+1\right)\right)}{F\left(\Pr[L_\theta]\right)}.\end{aligned}$$

As we have assumed by contradiction that $\Pr[L_\theta]/\Pr[L_{\theta^*}] \not\to 0$, $\left(\frac{\Pr[L_{\theta^*}]}{\Pr[L_\theta]}+1\right)$ has a strictly positive, finite limit. By Assumption 1, the above expression must have a finite limit, meaning that $t_n^*/t_n \not\to \infty$. □

For any $\theta \in (0,1)$, Lemma 4 indicates that $U_n(P)$ is equal to $\Pr[L_\theta]$ times $r(P)$, which is finite. Hence Lemma 8 implies

$$\lim_n \frac{U_n(P)}{U_n^*(P)} = 0 \quad \text{for all } P \text{, for all } \theta \in (0,1) \setminus \{\theta^*\}.$$

To complete the proof of Theorem 3, we tie up the loose ends of the cases when $\theta = 0$ or 1. If the voting rule is dictatorial (i.e., $(\theta, \rho) = (0,1)$ or $(1,0)$), the voting benefit is zero for all n and Theorem 3 is trivially true. If $\theta = 0$ but $\rho < 1$, then $U_n(B) = 0$ for all n and $\Pr[L_\theta] = e^{-p_n(A)n}$. One can use a similar argument as in the proof of Lemma 8, noting

$$\begin{aligned}\lim_n \frac{\Pr[L_\theta]}{\Pr[L_{\theta^*}]} &= e^{-p_n(A)n} z^{\theta^*}(T)\sqrt{t_n^* n\theta^*(1-\theta^*)} \\ &= e^{-p_n(A)n} z^{\theta^*}(T)\sqrt{\frac{t_n^*}{\alpha(A)p_n(A)}\alpha(A)p_n(A)n\theta^*(1-\theta^*)}.\end{aligned}$$

Using $t_n = \alpha(A)p_n(A)$ when $\theta = 0$, proceed as in the proof of Lemma 8. The case for $\theta' = 1$ but $\rho' > 0$ is similar. □

Proof of Corollary 2. When $\alpha(A) = 1/2$, any voting rule gives the same expected benefit from abstaining. Thus the ex-ante expected social payoff is maximized when the expected net marginal benefit from voting is. Collarary 2 then follows from Theorem 3 and that $\theta^* = 1/2$ when $\alpha(A) = 1/2$. □

Proof of Proposition 4. Note that the turnout rate is given by

$$t_n = \sum_{P \in \{A,B\}} \alpha(P) F\left(U_n\left(P\right)\right).$$

The proof of Theorem 3 indicates that, for any $\theta \neq \theta^*$, there exists an N such that, for *both* P, $U_n^*(P) > U_n\left(P\right)$ for all $n > N$. Thus $t_n^* > t_n$ for all such n as well. □

Appendix E. Electoral Regimes

This appendix collects the proofs of theorems in Section 5. For this purpose, in this appendix, let $\overline{U}_n^s(P)$ and $U_n^s(P)$ be the abstaining benefit and the gross marginal benefit of voting, respectively, to a P-supporter when there is a subsidy of $s \geq 0$. The case of $s = 0$ corresponds to no subsidy, while $s = 1$ (the highest possible voting cost) corresponds to compulsory voting.

The next fact ensures that we can use the probability and payoff approximations in Appendix B.2 even when voting is subsidized.

Fact 7. Given a non-dictatorial voting rule, for any $s > 0$ and for $P = A, B$, $U_n^s(P) \to 0$ as $n \to \infty$.

Proof. When $s > 0$, a strictly positive fraction of citizen (from each party) will be voting. Hence t_n goes to a strictly positive limit and therefore $T_n \to \infty$. In addition, the limit of $\tau_n(A)/\tau_n(B)$ must be strictly positive and bounded. Hence β is bounded.

The above also mean that we can apply Lemma 3 to approximate $\Pr\left[L_\theta\right]$ even when $s > 0$. Since $m_n \leq 0$ and $T_n \to \infty$, It is immediately that $\Pr\left[L_\theta\right] \to 0$. As β is bounded, so is $r(P)$. By Lemma 4, $U_n^s(P) \to 0$ for each P. □

In the rest of this appendix, we will first prove all the results for subsidized and compulsory voting, and then prove the results for random decision-making.

E.1. **Subsidized and Compulsory Voting.**

Proof of Theorem 4. Given a subsidy level s, a (θ, ρ)-rule and an n, write the ex-ante expected payoff to a representative citizen as

$$W_n^s(\theta, \rho) = \sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n^s(P) + F\left(U_n^s(P) + s\right) U_n^s(P) - \int_0^{U_n^s(P)+s} cf(c)\, dc\right]$$

$$= \sum_{P \in \{A,B\}} \alpha(P) \Bigg[\overline{U}_n^s(P) + F\left(U_n^s(P) + s\right) \left[U_n^s(P) + s\right]$$

$$- \int_0^{U_n^s(P)+s} cf(c)\, dc - F\left(U_n^s(P) + s\right) s \Bigg]. \qquad (13)$$

There are two cases to be considered separately: $s < 1$ or $s \geq 1$. If $s < 1$, by Fact 7, $\lim_n U_n^s(P) + s < 1$ for both P. Using Fact 6, one can rewrite Equation (13) as

$$W_n^s(\theta, \rho) = \sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n^s(P) + \int_0^{U_n^s(P)+s} F(c) dc - F\left(U_n^s(P) + s\right) s \right].$$

Hence given a (θ, ρ)-rule and an $s > 0$,

$$\lim_{n \to \infty} W_n^0(\theta, \rho) - W_n^s(\theta, \rho)$$
$$= \lim_{n \to \infty} \sum_{P \in \{A,B\}} \alpha(P) \left\{ \left[\overline{U}_n^0(P) - \overline{U}_n^s(P) \right] - \int_{U_n^0(P)}^{U_n^s(P)+s} F(c)\, dc + F\left(U_n^s(P) + s\right) s \right\}. \qquad (14)$$

When $\alpha(A) \geq 1/2$ and $\theta < \theta^*$, by Theorem 1,

$$\sum \alpha(P) \overline{U}_n^0(P) = \alpha(A),$$

which is the maximum possible abstaining benefit. Thus

$$\lim_{n \to \infty} \sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n^0(P) - \overline{U}_n^s(P) \right] \geq 0.$$

Meanwhile, due to Fact 7, for each P,

$$\lim_n F\left(U_n^s(P) + s\right) s - \int_{U_n^0(P)}^{U_n^s(P)+s} F(c)\, dc = F(s)s - \int_0^s F(c)\, dc > 0,$$

since $F(s) > F(c)$ for all $c < s$. Hence the limit in Equation (14) is strictly positive.

If $s \geq 1$, Equation 13 can be rewritten as

$$\sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n^s(P) + \left[U_n^s(P) + s\right] - \int_0^1 cf(c)\, dc - s \right]$$
$$= \sum_{P \in \{A,B\}} \alpha(P) \left[\overline{U}_n^s(P) + U_n^s(P) - 1 - \int_0^1 F(c)\, dc \right]$$

Thus given a (θ, ρ)-rule and an $s > 0$,

$$\lim_n W_n^0(\theta, \rho) - W_n^s(\theta, \rho)$$
$$= \lim_n \sum_{P \in \{A,B\}} \alpha(P) \left\{ \left[\overline{U}_n^0(P) - \overline{U}_n^s(P) \right] + \int_0^{U_n^0(P)} F(c)\, dc - U_n^s(P) + 1 - \int_0^1 F(c)\, dc \right\}. \qquad (15)$$

Again, the difference in the expected abstaining payoffs is weakly positive. Meanwhile, due to Fact 7, for each P,

$$\lim_n \int_0^{U_n^0(P)} F(c)\, dc - U_n^s(P) + 1 - \int_0^1 F(c)\, dc = 1 - \int_0^1 F(c)\, dc > 0,$$

since $F(c) < 1$ for all $c < 1$. Hence the expression in Equation (15) must be strictly positive. □

Proof of Corollary 3. Apply Theorem 4, using $s = 1$ (the highest possible voting cost). □

Proof of Proposition 5. Take any $\rho, \rho^* \in (0,1)$. Abusing notations, write $\alpha = \alpha(A)$ for this proof. Using Equation (15),

$$\begin{aligned}
&\lim_{n\to\infty} W_n^0(\theta^*, \rho^*) - W_n^1(\alpha, \rho) \\
&\quad = \lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P) \Bigg\{ \Big[\overline{U}_n^0(P;\theta^*) - \overline{U}_n^1(P;\alpha) \Big] \\
&\qquad\qquad + \int_0^{U_n^0(P;\theta^*)} F(c)dc - U_n^1(P;\alpha) + 1 - \int_0^1 F(c)\, dc \Bigg\}.
\end{aligned}$$

Under compulsory voting with a threshold of α,

$$\beta = \frac{1-\alpha}{\alpha} \frac{\alpha}{1-\alpha} = 1.$$

By Lemma 6, then

$$\lim_{n\to\infty} \Pr[A \text{ wins}] = \frac{1}{2},$$

which is the same as under voluntary voting with a threshold of θ^*. Therefore

$$\lim_{n\to\infty} \sum_{P\in\{A,B\}} \alpha(P) \Big[\overline{U}_n^0(P;\theta^*) - \overline{U}_n^1(P;\alpha) \Big] = 0.$$

Meanwhile, by the same argument as in the proof of Theorem 4,

$$\lim_n \int_0^{U_n^0(P;\theta^*)} F(c)\, dc - U_n^s(P;\alpha) + 1 - \int_0^1 F(c)\, dc > 0.$$

Hence $\lim_n W_n^0 - W_n^s > 0$. □

Proof of Corollary 4. Apply Proposition 5, noting that $\theta^*(\alpha) = 1/2$ when $\alpha(A) = 1/2$. □

E.2. **Random Decision-making.** Consider a random decision-making regime electing A with probability $q \in (0,1)$. (Flipping a fair coin corresponds to $q = 1/2$ while a random dictatorship corresponds to $q = \alpha(A)$.) Denote the ex-ante expected social payoff from this regime as

$$W^q = \alpha(A)q + \alpha(B)(1-q).$$

Consider a non-dictatorial (θ, ρ)-rule. For any n,

$$W_n(\theta,\rho) - W^q$$
$$= \alpha(A)\left(\overline{U}_n(A) - q\right) + \alpha(B)\left(\overline{U}_n(B) - (1-q)\right) + \sum_{P\in\{A,B\}} \alpha(P) \int_0^{U_n(P)} F(c)\, dc.$$

We will use this expression for the next two proofs.

Proof of Theorem 5. When $\theta < \theta^*$, by Proposition 2,

$$\lim_n \overline{U}_n(A) = 1 \qquad \text{and} \qquad \lim_n \overline{U}_n(B) = 0.$$

Thus

$$\lim_{n\to\infty} W_n(\theta,\rho) - W^q = \big(\alpha(A) - \alpha(B)\big)(1-q) > 0. \qquad \square$$

Proof of Proposition 6. When $\alpha(A) = 1/2$,

$$W_n(\theta,\rho) - W^q = \frac{1}{2}\left(\overline{U}_n(A) - q\right) + \frac{1}{2}\left(\overline{U}_n(B) - (1-q)\right) + \frac{1}{2}\sum_{P\in\{A,B\}} \int_0^{U_n(P)} F(c)\, dc$$
$$= \frac{1}{2}\sum_{P\in\{A,B\}} \int_0^{U_n(P)} F(c)\, dc,$$

which is strictly positive for all finite n. $\square$

Appendix F. Relationship to Krishna and Morgan (2015)

Krishna and Morgan (2015) address the optimality of simple majority in terms of its ability to elect the utilitarian choice. This appendix describes how the Krishna-Morgan model can be translated into our set up, and vice versa. We also prove Theorem 2 of Krishna and Morgan (2015) in our setting to demonstrate the relationship between the two models.

In Krishna and Morgan, if citizen i is a P-supporter, she receives a private benefit of v_i^P when P is elected. Conditional on P, v_i^P is i.i.d. across citizens and is drawn from $[0,1]$ according to cumulative distribution G_P. Meanwhile, citizen i's private cost of voting, c_i, is drawn from distribution F on $[0,1]$ with $f(0) > 0$. The cost of voting is i.i.d. across citizens and is independent of party preferences. For each citizen, the benefit and cost distributions are independent of each other.

Krishna and Morgan call a choice P *utilitarian* if it is the ex-ante expected utility maximizer, that is, if it solves

$$\max_{P\in\{A,B\}} \alpha(P) \int_0^1 v\, dG_P(v).$$

(Note: Krishna and Morgan use λ for $\alpha(A)$.)

Given a Krisnha-Morgan model, normalize citizen i's cost and benefit by dividing both by (the realization of) v_i^P. This way all citizens receive a benefit of 1 if her preferred party is elected and have a normalized cost of voting given by c_i/v_i^P, a random variable whose distribution is asymmetric across parties. Specifically, given a party P, the c.d.f. of the voting cost distribution of P-supporter is given by

$$\begin{aligned} H_P(c) &= \Pr\left[\frac{c_i}{v_i^P} \leq c\right] \\ &= \int_0^1 \int_0^{cv} f(c_i) g_P(v)\, dc_i\, dv \\ &= \int_0^1 F(cv) g_P(v)\, dv. \end{aligned}$$

One can quickly check that H_P is a valid c.d.f. on $\mathbb{R}_+$: $H_P(c) \geq 0$ for all $c \geq 0$, the associated p.d.f. $h_P(c)$ is (strictly) positive for all $c \geq 0$, $H_P(0) = 0$ and $\lim_{c\to\infty} H(c) = 1$. In other words, a Krishna-Morgan model can be mapped into our set up with party-dependent voting cost distributions H_A and H_B given by the above equation.

Now suppose we are given a set up as in this paper with party-dependent voting cost distributions H_A and H_B. Assumption 1 is adjusted to the following:

Assumption 1′. For each P, and for any real number $x > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{H_P(\varepsilon x)}{H_P(\varepsilon)} \in (0, \infty) \text{ and is continuous in } x.$$

Provided that this party-dependent voting cost model is correctly specified (i.e., it is generated from a Krishna-Morgan model specified above), the utilitarian choice can be identified (without recovering F and the G_P's). To see this, note that, using L'Hôpital Rule,

$$\begin{aligned} \lim_{c\downarrow 0} \frac{\alpha(A) H_A(c)}{\alpha(B) H_B(c)} &= \lim_{c\downarrow 0} \frac{\alpha(A) \int_0^1 f(cv) v g_A(v)\, dv}{\alpha(B) \int_0^1 f(cv) v g_B(v)\, dv} \\ &= \frac{\alpha(A) \int_0^1 v g_A(v)\, dv}{\alpha(B) \int_0^1 v g_B(v)\, dv} \end{aligned}$$

(since $f(0) > 0$). Therefore, A is the utilitarian choice if this limit is larger than 1; and B if the limit is smaller than 1.

In such a model, redefine

$$\begin{aligned} \tau_n(P) &= \frac{\alpha(P) H_P(U_n(P))}{t_n} \\ \beta_n &= \frac{\tau_n(A)}{\tau_n(B)} \frac{1-\theta}{\theta} = \frac{\alpha(A) H_A(U_n(A))}{\alpha(B) H_B(U_n(B))} \frac{1-\theta}{\theta}. \end{aligned}$$

Note that all the approximations in Appendix B, hence Lemma 1 and Proposition 1, remain valid after this modification. So do Lemma 5 and 6 in the proof of Proposition 2. Fact 5 (in the proof of Proposition 2) can be easily modified into

Fact 5′. Let $\{\varepsilon_n\}$ and $\{\varepsilon'_n\}$ be two sequences of strictly positive numbers converging to zero. Suppose Assumption 1′ holds. Then

$$\lim_{n\to\infty} \frac{H_A\big(\varepsilon_n(1-\theta)\big)}{H_B\big(\varepsilon_n\theta\big)} \leq \lim_{n\to\infty} \frac{H_A\big(\varepsilon'_n(1-\theta')\big)}{H_B\big(\varepsilon'_n\theta'\big)}$$

whenever $\theta \geq \theta'$.

The proof is exactly the same as that of Fact 5 (only changing the F to the H_P's) and is therefore omitted. Lemma 7 now becomes

Lemma 7′. *There exists a unique $\theta^* \in (0,1)$ (which depends on α and the H_P's) such that*

$$\begin{aligned} \beta > 1 &\quad \textit{whenever } \theta < \theta^* \\ \beta = 1 &\quad \textit{whenever } \theta = \theta^* \\ \beta < 1 &\quad \textit{whenever } \theta > \theta^*. \end{aligned}$$

Proof. First we show that there is a unique θ^* such that $\beta^* = 1$. As in the proof of Lemma 7, we are looking for the solution to

$$\lim_n \frac{\alpha(A)H_A\left(s_n(1-\theta)\right)}{\alpha(B)H_B\left(s_n\theta\right)} = \frac{\theta}{1-\theta}. \tag{9'}$$

By Fact 5′, the left hand side is weakly decreasing in θ. Meanwhile, the right hand side is strictly increasing in θ. When $\theta = 0$, the left hand side approaches infinity while the right hand side is zero. When $\theta = 1$, the left hand side is zero while the right hand side approaches infinity. Since both sides are continuous functions of θ, there exists a unique $\theta^* \in (0,1)$ solving Equation (9′).

The second part of the proof — showing that $\beta > 1$ whenever $\theta < \theta^*$ (and $\beta < 1$ whenever $\theta > \theta^*$) is the same as in the proof of Lemma 7, using Fact 5′ instead of Fact 5. □

Compared to Lemma 7, Lemma 7′ establishes the existence of θ^* only in $(0,1)$, rather than the more precise interval $[1/2, \alpha(A)]$. Also, the value of θ^* cannot be known even if $\alpha(A) = 1/2$. This is due to the fact that θ^* now depends on the H_P's functions as well. Nevertheless, for any threshold θ, the size of β relative to 1 can be determined by the size of θ relative to θ^*. This leads us to the counter-part of Proposition 2:

Proposition 2′. *There exists a unique $\theta^* \in (0,1)$ such that*

$$\lim_{n\to\infty} \Pr\left[A \textit{ wins} \mid n\right] = \begin{cases} 1 & \textit{if } \theta < \theta^* \\ 1/2 & \textit{if } \theta = \theta^* \\ 0 & \textit{if } \theta > \theta^* \end{cases}.$$

Proof. This follows from Lemmas 5, 6 and 7′. □

Proposition $2'$ is not only an extension of Proposition 2, but also a key to proving Krishna and Morgan's theorem in our setting, which can be stated as follows:

Theorem 6 (Krishna and Morgan, 2015, Theorem 2)**.** *Under voluntary voting, a* (θ, ρ)*-rule elects the utilitarian choice with probability* 1 *at the limit for each and every* α*,* H_A *and* H_B *combination if and only if* $\theta = 1/2$*.*

The proof of this theorem makes use of the following lemma:

Lemma 9. *Given* α*,* H_A *and* H_B*, for any threshold* θ*,*

$$\frac{\alpha(A)\int_0^1 v g_A(v)\,dv}{\alpha(B)\int_0^1 v g_B(v)\,dv} > \left(\frac{\theta}{1-\theta}\right)^2 \quad \text{if } \theta < \theta^*$$
$$\frac{\alpha(A)\int_0^1 v g_A(v)\,dv}{\alpha(B)\int_0^1 v g_B(v)\,dv} = \left(\frac{\theta}{1-\theta}\right)^2 \quad \text{if } \theta = \theta^*$$
$$\frac{\alpha(A)\int_0^1 v g_A(v)\,dv}{\alpha(B)\int_0^1 v g_B(v)\,dv} < \left(\frac{\theta}{1-\theta}\right)^2 \quad \text{if } \theta > \theta^*$$

Proof. First suppose $\theta < \theta^*$. By Lemma $7'$, $\beta > 1$. Hence

$$\begin{aligned}
\lim_{n\to\infty} \frac{\alpha(A)H_A(U_n(A))}{\alpha(B)H_B(U_n(B))} &> \frac{\theta}{1-\theta} \\
\lim_{\Pr[L_\theta]\to 0} \frac{\alpha(A)H_A(\Pr[L_\theta]\,r(A))}{\alpha(B)H_B(\Pr[L_\theta]\,r(B))} &> \frac{\theta}{1-\theta} && \text{(Lemma 4)} \\
\frac{r(A)\alpha(A)\int_0^1 v g_A(v)\,dv}{r(B)\alpha(B)\int_0^1 v g_B(v)\,dv} &> \frac{\theta}{1-\theta} && \text{(L'Hôpital Rule)} \\
\frac{\alpha(A)\int_0^1 v g_A(v)\,dv}{\alpha(B)\int_0^1 v g_B(v)\,dv} &> \frac{r(B)}{r(A)}\frac{\theta}{1-\theta} \\
&\geq \left(\frac{\theta}{1-\theta}\right)^2. && \text{(Remark 3)}
\end{aligned}$$

The proofs for the other two cases are similar and are therefore omitted. □

Proof of Theorem 6.

If: Using Proposition $2'$, when $\theta = 1/2$, Lemma 9 implies

$$\lim_n \Pr[A \text{ wins} \mid n] = 1 \quad \text{if} \quad \frac{\alpha(A)\int_0^1 v g_A(v)\,dv}{\alpha(B)\int_0^1 v g_B(v)\,dv} > 1$$
$$\lim_n \Pr[A \text{ wins} \mid n] = \frac{1}{2} \quad \text{if} \quad \frac{\alpha(A)\int_0^1 v g_A(v)\,dv}{\alpha(B)\int_0^1 v g_B(v)\,dv} = 1$$
$$\lim_n \Pr[A \text{ wins} \mid n] = 0 \quad \text{if} \quad \frac{\alpha(A)\int_0^1 v g_A(v)\,dv}{\alpha(B)\int_0^1 v g_B(v)\,dv} < 1,$$

meaning that the utilitarian choice is elected with probability 1.

Only if: Suppose $\theta > 1/2$. If

$$\left(\frac{\theta}{1-\theta}\right)^2 > \frac{\alpha(A)\int_0^1 v g_A(v)\, dv}{\alpha(B)\int_0^1 v g_B(v)\, dv} > 1,$$

by Proposition 2′ and Lemma 9, B wins with probability 1 at the limit even though A is the utilitarian choice. Similarly, when $\theta < 1/2$, if

$$1 > \frac{\alpha(A)\int_0^1 v g_A(v)\, dv}{\alpha(B)\int_0^1 v g_B(v)\, dv} > \left(\frac{\theta}{1-\theta}\right)^2,$$

then A wins with probability 1 even though B is the utilitarian choice. □

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