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Welfare comparison of electoral systems under power sharing

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Abstract We generalize Börgers' (Am Econ Rev 94:57–66, 2004) results to a broad class of power sharing electoral systems. We show that voluntary voting under a power sharing regime Pareto dominates both random decision making and compulsory voting. We also show, however, that voluntary voting is not socially optimal, as individuals vote too frequently.

1 Introduction

Should voting be voluntary or compulsory, when it is costly? Börgers (2004) answers these questions for the case of simple majority (i.e., winner-take-all) rule and an evenly split electorate. Our paper generalizes Börgers' results to a broad class of voting systems under power sharing (i.e., the minority gets a share of the benefits), noting that most electoral systems are not winner-take-all. This can be for two reasons. Firstly, the regime could formally be a power sharing one. For examples, the electoral rule may be fully or partly proportional; or the legislative system has in-built separation

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of powers, checks and balances, parliamentary committees, veto powers, etc. that guarantee minority representation even if the electoral rule is winner-take-all (see Herrera et al. 2014).[1] Secondly, even in systems which are formally winner-take-all, mandate can significantly influence the policy implemented by the winner. Based on U.S. Congressional voting records, Faravelli et al. (2015) show that the policies adopted by elected politicians depend on the margin of victory, with congressmen being more likely to behave moderately when their margin of victory is small than when they win by a landslide.

An intuitive way of thinking about power sharing regimes is to consider two parties who have to split a cake. Under majority rule, the winner takes it all. Under a power sharing rule, the share of the cake a party secures is a weakly increasing function of the proportion of votes it obtains. We impose only two innocuous assumptions on the shape of the function. First, anonymity, i.e., the two parties are equally treated independently of their labels. Second, it is weakly more beneficial for the minority to close the gap with the majority than for the majority to increase it. Notice that majority rule fulfills both properties and thus is a special case of our class of power sharing rules.

We show that, when the electorate is evenly split, voluntary voting under a power sharing regime Pareto dominates both random decision making and compulsory voting. We also show, however, that voluntary voting is not socially optimal. Quite surprisingly, this is not because individuals free ride on each other and thus vote too little, but because too many people turn out to vote.

Börgers (2004) explains the superiority of voluntary voting over compulsory voting by the negative externality an extra vote imposes onto the pivotal probabilities of the other voters. Under power sharing rules, there is no pivotal vote. However, we demonstrate a similar externality on the impact that other votes have on the election outcome. The comparison between voluntary voting and random decision making is also more delicate under power sharing rules, as the random decision process needs to determine not only the winner, but also the margin of victory. As the electorate is evenly split, it is tempting to believe that splitting the cake equally and saving the cost of voting ought to be socially preferable to voluntary voting. This intuition is fallacious because those who choose to abstain under voluntary voting receive the same benefit as under random decision making. Those who decide to vote when voting is voluntary must be receiving a higher expected benefit from voting than from abstaining, as they would abstain otherwise. Thus, as long as someone turns out to vote with positive probability, social welfare under any power sharing rule must be higher than under random decision making.

2 Model

There are $N + 1$ individuals and two alternatives, A and B. A generic individual will be denoted i and a generic alternative will be denoted P, with its opponent being Q.

[1] Castanheira (2003), Kartal (2015) and Faravelli and Sanchez-Pages (2015), among others, have also investigated the participation and welfare implications of power sharing.

Ex-ante, each individual independently strictly prefers A to B with probability $1/2$ and strictly prefers B to A with the same probability. Individual i bears a voting cost of c_i if she votes. Voting costs of individuals are identically and independently drawn according to a cumulative distribution function F from a non-degenerate support $C \subset \mathbb{R}_+$. We assume that F has a probability density function f such that $f(c) > 0$ for all $c \in C$. Each individual's preference and voting cost are her private information, but their distributions are commonly known.

We consider three voting systems:

Voluntary voting Each individual chooses simultaneously to vote for A, B or to abstain.
Compulsory voting Each individual chooses simultaneously to vote for A or B.
Random decision making No individual votes. $N+1$ random votes will be independently generated. Each random vote is cast for each alternative with probability $1/2$.

Regardless of how the votes are generated, the *mandate* of an alternative is defined as the proportion of the (random) votes it receives, except when no one votes (under voluntary voting), in which case each alternative receives a mandate of $1/2$. Formally, let n_P and n_Q be the total number of votes cast for P and Q, respectively. The mandate for alternative $P \in \{A, B\}$ is

$$m_P(n_P, n_Q) = \begin{cases} \frac{n_P}{n_P+n_Q} & \text{if } n_P + n_Q > 0 \\ \frac{1}{2} & \text{if } n_P + n_Q = 0 \end{cases}.$$

If the favorite alternative of an individual receives a mandate of m, this individual gets a mandate benefit of $G(m)$ where $G : [0, 1] \to [0, 1]$ is a weakly increasing function that is symmetric around $1/2$—that is, $G(m) = 1 - G(1-m)$ for each $m \in [0, 1]$. Normalize the mandate benefit such that $G(0) = 0$ and $G(1) = 1$. Further assume that G is weakly concave on $[1/2, 1]$. Symmetry of G around $1/2$ and weak concavity of G above $1/2$ imply that G is weakly convex on $[0, 1/2]$. Thus, conditional on the preferred alternative winning a majority, the returns to mandate is diminishing. In voting games, it also means that an extra vote is worth more to the losing party than to the winning party. Note that G can be discontinuous at $1/2$—for instance, when there is an extra benefit to winning.

The payoff to each individual is her benefit from mandate net of her voting cost if she has voted (voluntarily or otherwise). Consider an individual i who prefers alternative P with a voting cost of c_i. Let v_P and v_Q be the number of votes cast for P and Q, both exclusive of i. Then i's payoff is

$$\begin{aligned} G\big(m_P(v_P+1, v_Q)\big) - c_i \quad & \text{if } i \text{ votes for } P; \\ G\big(m_P(v_P, v_Q+1)\big) - c_i \quad & \text{if } i \text{ votes for } Q; \text{ and} \\ G\big(m_P(v_P, v_Q)\big) \quad & \text{if } i \text{ abstains.} \end{aligned}$$

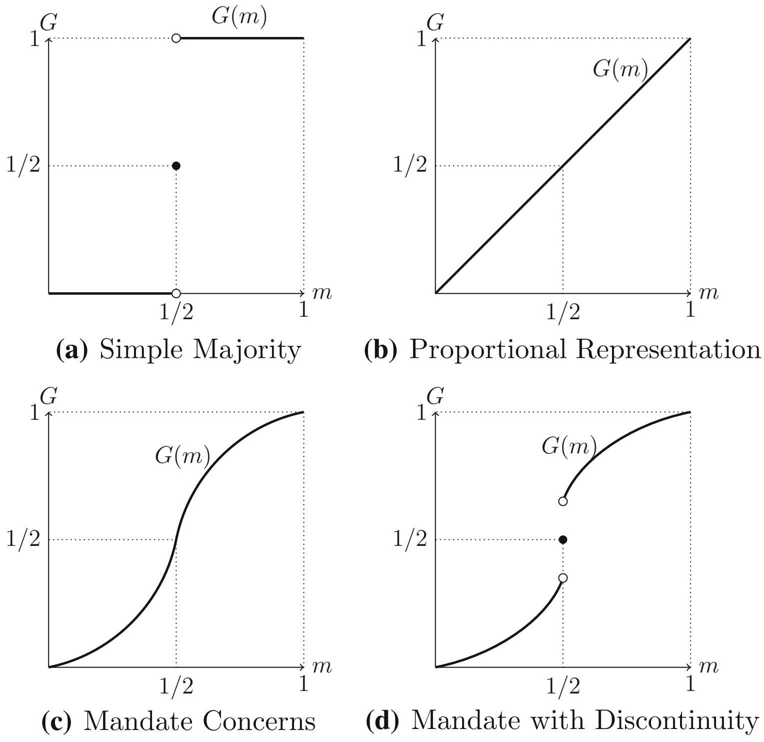


Fig. 1 Examples of admissible G function

There are many interpretations of the mandate benefit function G. The standard simple majority rule is the special case when G is the step function

$$G(m) = \begin{cases} 0 & \text{if } m < \frac{1}{2} \\ \frac{1}{2} & \text{if } m = \frac{1}{2} \\ 1 & \text{if } m > \frac{1}{2} \end{cases} \tag{1}$$

as depicted in Fig. 1a. The function G may depict different electoral rules. For example, an ideal proportional representation[2] can be represented by the specification $G(m) = m$ (see Fig. 1b). The G function can also accommodate the effect of mandate on policy outcomes Faravelli et al. 2015, power sharing Castanheira 2003; Herrera et al. 2014 and behavioral preferences for mandate (see Fig. 1(c)). Since G need not be continuous, it can also allow for concerns for mandate with a jump in payoff when the mandate is $1/2$ (see Fig. 1(d)).

As the electoral rule may depend on the mandate, rather than just the identity of the winner, we cannot use a simple coin toss (as in Börgers 2004) as our random decision making regime. Thus we require $N+1$ random votes to be generated. The mandate of

[2] "Ideal" meaning that seats are infinitely divisible.

an alternative can then be calculated as the proportion of random votes it obtains. This allows us to evaluate an individual's payoff using the G function. Notice that Börgers' coin toss is a special case of our random votes regime since, under the simple majority rule, an individual gets a payoff of 1 if her preferred alternative gets more than half of the votes—which occurs with probability $1/2$ if each random vote is cast for either alternative with probability $1/2$.

3 Strategic voting behavior

3.1 Compulsory voting

A compulsory voting regime defines a Bayesian game in which a pure strategy for an individual is a measurable voting rule $s_i : \{A, B\} \times C \to \{A, B\}$ assigning an alternative choice for each preference and cost combination. However, regardless of the individual's voting cost and the other individuals' strategy profiles, voting for the less preferred alternative is always weakly and sometimes strictly worse than voting for the preferred alternative. Therefore, each individual always votes for her preferred alternative in the unique Bayesian-Nash equilibrium in undominated strategy.

3.2 Voluntary voting

A pure strategy in a Bayesian game defined by a voluntary voting regime can be described by a pair of measurable voting rules (s_i^A, s_i^B) where $s_i^P : C \to \{A, B, \text{Abstain}\}$ assigns a choice for each voting cost when i prefers alternative P.

Notice that for each preference type and voting cost realization, voting for the less preferred alternative is always worse than abstaining regardless of the votes cast by other individuals. (It is strictly worse unless the realized voting cost is exactly zero, which is a zero probability event.) In addition, given individual i's preference and other individuals' voting rules, the difference between the payoffs from voting for i's preferred alternative and abstaining is strictly decreasing in i's cost of voting. Thus any best response of i to any opponent strategy profile must be monotone—i.e., for each alternative P, there exists a cut-off cost c_i^P such that when i prefers P, she votes for P for all $c_i < c_i^P$ and abstains for all $c_i > c_i^P$.

We consider (ex-ante) symmetric Bayesian-Nash equilibria in which each individual adopts the same pair of voting rules (s^A, s^B). This requirement by itself does not guarantee type-symmetry—that is, supporters of both parties vote with the same probability. Nonetheless, as the two alternatives are ex-ante symmetric, one would expect the same behavior from the supporters for the two alternatives in equilibrium, as in the following lemma.

Lemma 1 *Each individual, regardless of her preference, votes with the same probability in any symmetric equilibrium of the voluntary voting game. That is, if* (c^A, c^B) *is a pair of cut-off costs associated with a symmetric equilibrium strategy* (s^A, s^B)*, then* $F(c^A) = F(c^B)$.

Proof This follows from Proposition 1 of Kartal (2015) (with $p = 1/2$) as our G function is a regular electoral system by Kartal's definition. □

In Börgers (2004), type-symmetry is an assumption. Lemma 1 indicates that type-symmetry follows from our set up, which encompasses Börgers'.

Given the above argument, a symmetric Bayesian-Nash equilibrium can be described by a single turnout rate p^* such that an individual with cost $c = F^{-1}(p^*)$ is indifferent between voting and abstaining.

Proposition 1 *The voluntary voting game has a unique symmetric equilibrium voting probability.*[3]

Proposition 1 of Börgers (2004) is a special case of this proposition, since the step function G in Eq. (1) satisfies all our assumptions.

It is possible that p^*, the equilibrium voting probability, is 0 or 1. However, since the expected gross marginal benefit from voting (i.e., benefit from voting over that from abstaining, without subtracting the voting cost) is $1/2$ when $p = 0$, if the lower bound of the support of voting cost is strictly less than $1/2$, then $p^* > 0$ in any symmetric equilibrium. Similarly, since the expected gross marginal benefit from voting is bounded above by $1/2$, if the upper bound of the support of voting cost is higher than $1/2$, then $p^* < 1$.

Our proof of the existence of a type-symmetric Bayesian-Nash equilibrium of the voluntary voting game does not invoke the concavity assumption on G (see Appendix). However, without the concavity assumption, we cannot guarantee that every symmetric equilibrium is type-symmetric, nor can we guarantee the uniqueness of the type-symmetric voting probability.

4 Welfare comparison

4.1 Comparison between voluntary voting and random decision making

Ex-ante, the distribution of preferences and the distribution of votes generated by random decision making are the same. Since random decision making saves all voting costs, one may think that it is superior to voluntary voting. The next proposition says that this "intuition" is false.

Proposition 2 *Voluntary voting interim Pareto dominates random decision making whenever the equilibrium voluntary voting probability is non-zero.* (*That is, voluntary voting yields a weakly higher interim expected payoff for each preference and voting cost realization, with a strictly higher interim expected payoff for at least one preference and cost realization.*)

The above mentioned "intuition" is false because the expected payoff under random decision making is the same as the expected payoff under voluntary voting *only when the individual abstains*. Since those who choose to vote under voluntary voting must

[3] All omitted proofs are relegated to the Appendix.

be getting a higher benefit than abstaining (otherwise they would have abstained), their welfare is higher under voluntary voting than under random decision making.

Upon inspection, the proof of Proposition 2 does not require the concavity assumption on G. Thus Proposition 2 is a very general statement—it holds for all symmetric electoral rules and all voluntary equilibria under a given electoral rule.

4.2 Comparison between voluntary and compulsory voting

At the other extreme, one can consider compulsory voting, where each individual is forced to obtain the benefit from voting. However, the next proposition states that compulsory voting is also worse than voluntary voting.

Proposition 3 *Voluntary voting interim Pareto dominates compulsory voting whenever the equilibrium voluntary voting probability is strictly less than 1.*

Propositions 2 and 3 are generalizations of Börgers (2004, Proposition 2) to a general class of electoral rules. They show that Börgers' welfare comparison does not rely on majority voting, which is known to give high turnout and welfare under voluntary voting when the electorate is evenly split (c.f., Faravelli and Sanchez-Pages 2015; Herrera et al. 2014; Kartal 2015). The assumptions we impose on the G function (weak monotonicity, symmetry and weak concavity) are fairly permissive and are satisfied by many intuitive electoral rules.

4.3 Socially optimal turnout

If neither random decision making nor compulsory voting is socially optimal, is voluntary voting efficient? The answer is no. However, contrary to common perception, voluntary voting is inefficient not because individuals free-ride on each other and vote too little, but because they vote too much.

Proposition 4 *The ex-ante socially optimal symmetric turnout rate is weakly lower than p^*, the symmetric equilibrium turnout rate under voluntary voting.*

Voluntary voting is "excessive" due to a negative externality arising from congestion—an extra vote reduces other voters' expected impact, hence their expected benefits from voting (see Sect. 5 below). Börgers (2004) makes a similar observation by arguing that the probability of an individual being pivotal is decreasing in the turnout rate under a simple majority electoral rule. Proposition 4 generalizes it to a general class of electoral rules.

Proposition 4 is an ex-ante welfare statement. The ex-ante social optimum does not interim Pareto dominate the voluntary voting equilibrium. This is because individuals who would have voted under voluntary voting but not under the socially optimal rule are obtaining an (equilibrium) interim expected benefit above the benefit of abstaining, and strictly so unless their voting cost is exactly at the equilibrium cut-off cost.

Here the social planner can only choose a single turnout rate for the whole society to maximize ex-ante welfare. Yet, if the social planner can achieve an even higher welfare by asking different individuals to vote with different probabilities, our conclusion that the voluntary voting equilibrium is inefficient would still hold.

5 Voting externality and diminishing returns

The welfare results in the previous section, as well as the uniqueness of our voluntary voting equilibrium, depend on the fact that there is a negative externality to voting. That is, the expected gross benefit to voting is decreasing in the voting participation rate (Lemma 3, Appendix 7.1). This fact, in turns, relies on our symmetry and curvature assumptions on the G function.

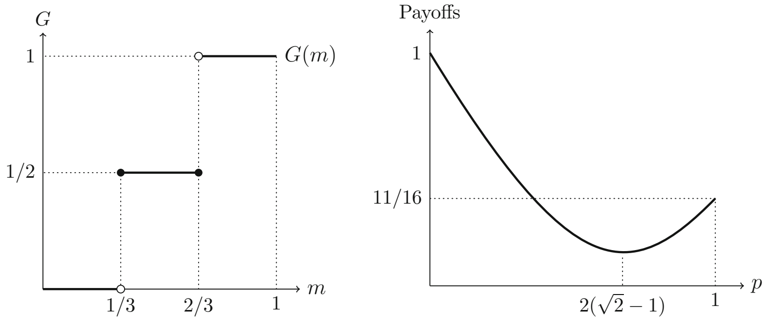
To see the role of the curvature assumption, consider an individual voter. Suppose there are M other voters who all vote with probability 1. (The expected benefit when others are voting with a probability less than 1 is just the expectation of voting benefit over M.) The preferences of these M other voters (which are random to this individual), together with her own vote, generate a distribution of the mandate of her favorite alternative on the support $\{1/(M+1), 2/(M+1), \ldots, 1\}$. When M increases, this support becomes finer. The probability mass on each of the points in the previous support is now distributed to the two nearby points in the new support. Considering these two nearby points and the distributed mass, the new expectation is an average of the payoff at the two nearby points, whereas the old expectation is the payoff at the point in the original support. On the interval $[1/2, 1]$, the payoff function is concave. By Jensen's inequality, had the point in the original support been the average (weighted by the distributed mass), the expectation of the payoff function would have already been lower than the payoff at the point in the original support. But on $[1/2, 1]$, the point in the original support is even *higher* than the weighted average of the two nearby points in the new support, as the distribution is becoming more concentrated at $1/2$. Since the payoff function is weakly increasing, the two effects reinforce each other, leading to a lower expected payoff conditional on the mandate being in $[1/2, 1]$. With a similar argument, the (conditional) expected payoff on $[0, 1/2]$ increases when M increases. As this individual is voting, her favorite wins with a probability greater than $1/2$. Thus she gets a lower expected payoff more frequently (than getting the higher expected payoff) when more people turn out to vote.

When the curvature assumption is lifted, a number of the previous results may not hold, as the following example illustrates.

Example 1 Suppose G is given by

$$G(m) = \begin{cases} 0 & \text{if } m < 1/3 \\ 1/2 & \text{if } 1/3 \le m \le 2/3 \\ 1 & \text{if } m > 2/3 \end{cases} .$$

This function is weakly increasing in m, symmetric around $1/2$ but is not concave on $[1/2, 1]$ (see Fig. 2a). Using the formula for the expected gross benefit to voting (Eq. (3), Appendix), one can verify that, when $N = 3$, the expected gross benefit is strictly increasing in p for $p > 2(\sqrt{2} - 1) \approx 0.8284$.



(a) G Function **(b)** Expected Gross Benefit of Voting

Fig. 2 Example 1: the role of the curvature assumption

Further suppose that voting costs are distributed on [0, 1/2] according to the cumulative distribution function

$$F(c) = \begin{cases} \frac{32}{7}c & \text{if } 0 \leq c < 0.175 \\ 32c - \frac{24}{5} & \text{if } 0.175 \leq c < 0.18 \\ \frac{1}{8}c + \frac{15}{16} & \text{if } 0.18 \leq c \leq 0.5 \end{cases}.$$

In this case, there are three voluntary voting equilibria, occurring at $p = 0.7882$, 0.9516 and 0.9601. The uniqueness result in Proposition 1 no longer holds. Moreover, none of the three equilibria interim Pareto dominates compulsory voting. The smallest one does not because the equilibrium gross voting benefit there is 0.6724, which is less than 11/16, the gross voting benefit when $p = 1$ (i.e., under compulsory voting). The other two equilibria do not because they occur on the interval where the expected gross benefit of voting is increasing. In all three cases, those voting at the equilibrium obtains a higher payoff under compulsory voting. Hence Proposition 3 does not hold. As a matter of fact, the ex-ante expected welfare in each of them is *lower* than that under compulsory voting (see Table 1).[4] The ex-ante social optimal voting probability occurs at 0.4125, which is lower than each of the equilibrium voting probabilities. In this sense Proposition 4 still holds. Nonetheless, locally at the two larger equilibria, there is a *positive* externality of voting,[5] which goes against the underlying idea of Proposition 4.[6]

To see the role of curvature in this example, consider the case when there are 2 other voting individuals. If an individual votes, her favorite alternative receives a

[4] However, compulsory voting does not interim Pareto dominate these voluntary voting equilibria. Those who are not voting at these equilibria could have a higher payoff under voluntary voting as their voting cost may be too high.

[5] There is a local maximum at $p = 0.9809$ for the social planner problem, achieving an ex-ante welfare of 0.5767. See Appendix 7.6.

[6] It is not true that at least one equilibrium must occur at a point where there is negative externality of voting. We could have constructed the cost distribution function such that the only voluntary voting equilibrium occurs on the interval where the expected gross benefit of voting is increasing.

Table 1 Numerical results for Example 1

p	Gross voting benefit	Social welfare
Voluntary voting equilibria		
0.7882	0.6724	0.5680
0.9516	0.6797	0.5741
0.9601	0.6809	0.5753
Compulsory voting		
1	0.6875	0.5755
Ex-ante social optimum		
0.4125	0.7588	0.5882

mandate of $2/3$ with probability $1/2$ (the probability that the 2 other individuals vote for different alternatives). When there are 3 other voting individuals, this probability mass is, roughly speaking, spread to a mandate of $1/2$ and a mandate of $3/4$. As G is not concave, the expected payoff to voting goes up in this particular instance.[7]

The role of anonymity (i.e., symmetry) of the G function is more subtle. When the two alternatives are treated differently, we would not expect the same voting behaviors from the supporters of the two alternatives. Voluntary voting equilibrium would not be symmetric and a more general definition of equilibrium is required. Interested readers should consult Faravelli and Man (2014) for the behavioral and welfare implications of non-anonymous voting systems.

6 Conclusion

This paper generalizes Börgers (2004) by showing that, when the electorate is ex-ante evenly split, voluntary voting under any weakly increasing, anonymous and diminishing returns (conditional on winning) power sharing rule is Pareto superior to both random decision making and compulsory voting. However, we also showed that voluntary voting is not socially optimal: because of the negative externality associated with voting, too many votes are cast.

Our generalization maintains Börgers' assumption that the electorate is ex-ante evenly split. There are two reasons for such a choice. First, intuitively, a situation without a known majority makes voting most relevant and interesting. Second, theoretically, an evenly split electorate would be the equilibrium outcome from a classical Downsian-Hotelling model of political platform choice. Krasa and Polborn (2009) show that the welfare under voluntary voting is continuous in the expected size of the majority (α in their model) around $1/2$. As the welfare under compulsory voting is also continuous in the same variable, the welfare comparison between the two is unchanged for α around $1/2$. The comparison between voluntary voting and random

[7] It should be noted that the G function in this example could have been made continuous (by making the function linear on the ε-balls around $1/3$ and $2/3$) and the same qualitative results would remain. It is the curvature, not (dis)continuity, that drives this example.

decision-making is not as straight-forward as it may seem when there is a strict majority. Uniform randomization between the two alternatives is inappropriate as there is a strict majority. Neither would the expected size of the majority correspond to the voluntary voting expected winning probability, due to the presence of the underdog effect (see Faravelli and Man 2014). A full analysis of this problem would go beyond the scope of this paper.

7 Proofs

Throughout this appendix, write individual i's expected benefit from abstaining when each of the N other individuals votes with probability p as[8]

$$\overline{B}(p, N) = \sum_{M=0}^{N} \binom{N}{M} p^M (1-p)^{N-M} \sum_{v=0}^{M} \binom{M}{v} \left(\frac{1}{2}\right)^M G(m(v, M-v)). \quad (2)$$

Similarly, the expected gross benefit (i.e., benefit without subtracting the voting cost) from voting for the preferred alternative when each of the N other individuals votes with probability p can be written as

$$B^*(p, N) = \sum_{M=0}^{N} \binom{N}{M} p^M (1-p)^{N-M} \sum_{v=0}^{M} \binom{M}{v} \left(\frac{1}{2}\right)^M G\left(\frac{v+1}{M+1}\right). \quad (3)$$

We will first prove two Lemmas, and then use them to prove the Propositions.

7.1 Two useful lemmas

Lemma 2 $\overline{B}(p, N) = 1/2$ *for all* $p \in [0, 1]$ *and* $N \geq 0$.

Proof First note that $\overline{B}(p, N)$ does not depend on the alternative. Moreover, Eq. (2) indicates that $\overline{B}(p, N)$ is the expectation of G to a supporter of either alternative. Since $G(m_A) + G(m_B) = 1$ for any m_A (and $m_B = 1 - m_A$), the expectation of the sum of G (across alternatives) must be 1. This implies $\overline{B}(p, N) = 1/2$.[9] □

Lemma 3 B^* *is weakly decreasing in* p *for all* $N \geq 0$.

Proof Using Eq. (3), we can write

$$B^*(p, N) = \sum_{M=0}^{N} \binom{N}{M} p^M (1-p)^{N-M} B^*(1, M).$$

In other words, $B^*(p, N)$ is the expectation of $B^*(1, M)$ over M according to the binomial distribution with N trials and success probability p. Recall that a binomial

8 For notational convenience, we adopt the convention $0^0 = 1$ throughout this paper.

9 We thank an anonymous referee for suggesting this simple proof.

distribution with success probability p and n trials first order stochastically dominates that with success probability $p' < p$ and n trials. Therefore, to prove that B^* is decreasing in p, it suffices to show that $B^*(1, \cdot)$ is decreasing (with respect to M).

Using Pascal's formula, we can write

$$B^*(1, M+1) = \sum_{v=0}^{M+1} \binom{M+1}{v} \left(\frac{1}{2}\right)^{M+1} G\left(\frac{v+1}{M+2}\right)$$
$$= \sum_{v=0}^{M} \binom{M}{v} \left(\frac{1}{2}\right)^{M} \left[\frac{1}{2} G\left(\frac{v+1}{M+2}\right) + \frac{1}{2} G\left(\frac{v+2}{M+2}\right)\right].$$

Meanwhile,

$$B^*(1, M) = \sum_{v=0}^{M} \binom{M}{v} \left(\frac{1}{2}\right)^{M} G\left(\frac{v+1}{M+1}\right).$$

□

Thus we can write

$$B^*(1, M) - B^*(1, M+1) = \sum_{v=0}^{M} \binom{M}{v} \left(\frac{1}{2}\right)^{M} h_M(v),$$

where $h_M : \{0, \ldots, M\} \to [0, 1]$ is defined by

$$h_M(v) = G\left(\frac{v+1}{M+1}\right) - \left[\frac{1}{2} G\left(\frac{v+1}{M+2}\right) + \frac{1}{2} G\left(\frac{v+2}{M+2}\right)\right].$$

This h_M function has the following properties:

Fact 1 *For all* $v = 0, \ldots, M$, $h_M(M - v - 1) = -h_M(v)$.

Proof Using the definition of h_M,

$$h_M(M - v - 1) = G\left(\frac{M-v}{M+1}\right) - \left[\frac{1}{2} G\left(\frac{M-v}{M+2}\right) + \frac{1}{2} G\left(\frac{M-v+1}{M+2}\right)\right]$$
$$= 1 - G\left(\frac{v+1}{M+1}\right)$$
$$- \left(\frac{1}{2}\left[1 - G\left(\frac{v+2}{M+2}\right)\right] + \frac{1}{2}\left[1 - G\left(\frac{v+1}{M+2}\right)\right]\right)$$
$$= -G\left(\frac{v+1}{M+1}\right) + \left[\frac{1}{2} G\left(\frac{v+1}{M+2}\right) + \frac{1}{2} G\left(\frac{v+2}{M+2}\right)\right]$$
$$= -h_M(v).$$

□

Fact 2 *For* $v \leq M/2 - 1$, $h_M(v) \leq 0$.

Proof Since G is convex on $[0, 1/2]$, for $v \leq M/2 - 1$,

$$\begin{aligned} h_M(v) &= G\left(\frac{v+1}{M+1}\right) - \left[\frac{1}{2}G\left(\frac{v+1}{M+2}\right) + \frac{1}{2}G\left(\frac{v+2}{M+2}\right)\right] \\ &\leq G\left(\frac{v+1}{M+1}\right) - G\left(\frac{1}{2} \times \frac{v+1}{M+2} + \frac{1}{2} \times \frac{v+2}{M+2}\right) \\ &= G\left(\frac{v+1}{M+1}\right) - G\left(\frac{v+1}{M+1} + \frac{M-2v-1}{2(M+1)(M+2)}\right) \\ &\leq 0. \end{aligned}$$

The last inequality follows because $M - 2v - 1 > 0$ for $v \leq M/2 - 1$. □

Given our notations, $B^*(1, M) - B^*(1, M+1)$ can be written as

$$\sum_{v=0}^{\lfloor \frac{M}{2} \rfloor - 1} \binom{M}{v} \left(\frac{1}{2}\right)^M h_M(v) + \mathbb{1}\left[M \text{ odd}\right] \binom{M}{\frac{M-1}{2}} \left(\frac{1}{2}\right)^M h_M\left(\frac{M-1}{2}\right) + \sum_{v=\lceil \frac{M}{2} \rceil}^{M} \binom{M}{v} \left(\frac{1}{2}\right)^M h_M(v),$$

where $\mathbb{1}[\cdot]$ is the indicator function, which equals 1 if its argument is true and 0 otherwise; and the notations $\lfloor x \rfloor$ and $\lceil x \rceil$ denote, respectively, the largest integer smaller than or equal to and the smallest integer greater than or equal to x (i.e., the floor and ceiling of x). Notice that $(M-1)/2 = M - (M-1)/2 - 1$. Hence if M is odd, by Fact 1, $h_M((M-1)/2) = 0$ and the second term is zero. Also, we can change the summation variable of the last summation from v to $v' = M - v - 1$ (except for the $v = M$ term). Using Fact 1, then

$$B^*(1, M) - B^*(1, M+1) = \sum_{v=0}^{\lfloor \frac{M}{2} \rfloor - 1} \left[\binom{M}{v} - \binom{M}{v+1}\right] \left(\frac{1}{2}\right)^M h_M(v) + \left(\frac{1}{2}\right)^M h_M(M).$$

By Fact 2, $h_M(v) \leq 0$ for each v summed in the first term. Meanwhile, since $v < M/2$ for each v summed, $\binom{M}{v} < \binom{M}{v+1}$. Hence the first term must be positive. The second term is positive since

$$\begin{aligned} h_M(M) &= G\left(\frac{M+1}{M+1}\right) - \left[\frac{1}{2}G\left(\frac{M+1}{M+2}\right) + \frac{1}{2}G\left(\frac{M+2}{M+2}\right)\right] \\ &= \frac{1}{2}\left[G(1) - G\left(\frac{M+1}{M+2}\right)\right] \\ &\geq 0. \end{aligned}$$

Therefore, $B^*(1, \cdot)$ is decreasing (in M), implying that $B^*(\cdot, N)$ is decreasing (in p).

7.2 Proof of Proposition 1

Recall that p^* is a symmetric equilibrium voting probability under voluntary voting if and only if it solves

$$F\left(B^*(p, N) - \overline{B}(p, N)\right) = p.$$

The left hand side of this equation is continuous and, by Lemmas 2 and 3, decreasing in p. The right hand side is continuous and strictly increasing in p. At $p = 0$, the left hand side is weakly positive (since it is a probability) while the right hand side is zero. At $p = 1$, the left hand side is weakly less than 1 (since it is a probability) while the right hand side is 1. Hence there exists a unique p^* solving the equation, meaning that the voluntary voting game has a unique symmetric equilibrium voting probability.

7.3 Proof of Proposition 2

To each individual, random decision making gives the same expected interim payoff as abstaining in a voting game with $N+1$ other individuals who all vote with probability 1. The interim expected payoff under this regime is therefore $\overline{B}(1, N+1)$.

Given an equilibrium voluntary voting probability $p^* > 0$, let c^* be the associated equilibrium cut-off cost. Since $p^* > 0$, there exists $c \in C$ such that $c < c^*$.

An individual with voting cost $c > c^*$ abstains under voluntary voting, yielding an interim expected payoff of $\overline{B}(p^*, N)$. By Lemma 2, this is the same as the interim expected payoff under the random decision making regime.

An individual with voting cost c^* is indifferent between voting and abstaining under voluntary voting. Thus her interim expected payoff under voluntary voting is $\overline{B}(p^*, N)$, which by Lemma 2 is equal to the interim expected payoff under random decision making.

An individual with voting cost $c < c^*$ votes under voluntary voting, yielding an interim expected payoff of $B^*(p^*, N) - c$. Note that c^* is defined such that $B^*(p^*, N) - c^* \geq \overline{B}(p^*, N)$ (the inequality may be strict if $p^* = 1$). The right hand side of this inequality is equal to $\overline{B}(1, N+1)$ due to Lemma 2. Hence, $B^*(p^*, N) - c > \overline{B}(1, N+1)$ for all $c < c^*$.

Therefore, at the interim stage, all individuals are at least as well off under voluntary voting than under random decision making. Those with voting cost strictly below c^* are strictly better off under voluntary voting.

7.4 Proof of Proposition 3

To an individual with voting cost c, compulsory voting gives the interim payoff of $B^*(1, N) - c$.

Given an equilibrium voluntary voting probability $p^* < 1$, let c^* be the associated equilibrium cut-off cost. Since $p^* < 1$, there exists $c \in C$ such that $c > c^*$.

An individual with voting cost $c > c^*$ strictly prefers abstaining to voting under voluntary voting, yielding an interim expected payoff of $\overline{B}(p^*, N) > B^*(p^*, N) - c$. Under compulsory voting, she gets $B^*(1, N) - c$, which by Lemma 3 is weakly less than $B^*(p^*, N) - c$. Therefore, she is strictly worse off under compulsory voting.

An individual with voting cost $c = c^*$ is indifferent between voting and abstaining under voluntary voting, so her interim expected payoff must be $B^*(p^*, N) - c^*$. By Lemma 3, this is weakly higher than $B^*(1, N) - c^*$, the interim expected benefit under compulsory voting.

An individual with voting cost $c < c^*$ votes under voluntary voting and obtains an interim expected payoff of $B^*(p^*, N) - c$. By Lemma 3, this is weakly higher than $B^*(1, N) - c^*$, the interim expected benefit under compulsory voting.

Therefore, at the interim stage, all individuals are at least as well off under voluntary voting than under compulsory voting. Those with voting cost strictly above c^* are strictly better off under voluntary voting.

7.5 Proof of Proposition 4

Suppose the social planner chooses a symmetric cut-off cost c to maximize the representative agent's ex-ante payoff. Then the decision problem is

$$\max_{c \in C} \quad F(c)B^*(F(c), N) + (1 - F(c))\overline{B}(F(c), N) - \int_{\underline{c}}^{c} c' f(c') dc'.$$

Using Lemma 2, the first order condition yields

$$f(c)\left[B^*(F(c), N) - \overline{B}(F(c), N) - c + F(c)\frac{\partial B^*}{\partial p}\right] = 0.$$

Recall that $f(c) > 0$ for all $c \in C$. By Lemma 3, $\frac{\partial B^*}{\partial p} \leq 0$. Hence, at the socially optimal turnout level, we must have

$$B^*(F(c), N) - \overline{B}(F(c), N) \geq c.$$

Meanwhile, the voluntary equilibrium voting probability p^* satisfies

$$B^*(p^*, N) - \overline{B}(p^*, N) = F^{-1}(p^*).$$

Since B^* is weakly decreasing in p, this implies that the socially optimal $p = F(c)$ is weakly lower than p^*.

7.6 Workings for Example 1

Using Eq. (3), one can obtain, in this example

$$B^*(p, 3) = \frac{11}{16}p^3 + \frac{15}{8}p^2(1 - p) + \frac{9}{4}p(1 - p)^2 + (1 - p)^3.$$

Table 2 Social planner's problem numerical results

p	Social welfare	SOC
FOC points		
0.4125	0.5882	−
0.8335	0.5671	+
0.9809	0.5767	−
Boundary points		
0	0.5	
1	0.5755	
Kinked points		
0.8	0.5676	
0.96	0.5753	

Hence

$$\frac{\partial B^*}{\partial p} = \frac{3}{16}\left(p^2 + 4p - 4\right).$$

Thus $B^*(\cdot, 3)$ is strictly increasing on $p \in (2(\sqrt{2} - 1), 1]$.

By Lemma 2, the three voluntary voting equilibria (in terms of the voting probability p) are solutions to

$$B^*(p, 3) - \frac{1}{2} = F^{-1}(p).$$

The three equilibria are obtained numerically.

We solve the social planner's problem in terms of the voting probability (instead of cut-off voting cost), that is

$$\max_p \quad \frac{1}{2} + p\left[B^*(p, 3) - \frac{1}{2}\right] - \int_0^{F^{-1}(p)} c\, dF(c).$$

Ignoring the kinks in the distribution function, the FOC is

$$B^*(p, 3) - \frac{1}{2} - F^{-1}(p) + p\frac{\partial B^*}{\partial p} = 0;$$

while the second order derivative of the objective function is

$$2\frac{\partial B^*}{\partial p} + p\frac{\partial^2 B^*}{\partial p^2} - \frac{1}{f(F^{-1}(p))}.$$

Table 2 shows the ex-ante social welfare at the points where the FOC holds (one of them is a local minimum, as the SOC indicates), as well as at the boundary points and the kinked points of the distribution function.

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