The Important Thing Is Not (Always) Winning but Taking Part: Funding Public Goods with Contests

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Abstract

This paper considers a public good game with heterogeneous endowments and incomplete information affected by extreme free riding. I overcome this problem through the implementation of a deterministic contest in which several prizes may be awarded. I identify a monotone equilibrium, in which the contribution is strictly increasing in the endowment. I prove that it is optimal for the social planner to set the last prize equal to zero, but otherwise total expected contribution is invariant to the prize structure. Finally, I show that private provision *via* a contest Pareto-dominates public provision and is higher than the total contribution raised through a lottery.

1. Introduction

This paper looks at multiple prize contests as a way to overcome the free-riding problem. It is well known that the public good provision resulting from individual voluntary contributions is generally suboptimal, because of the incentive to free ride associated with positive externalities (e.g., see Bergstrom, Blume, and Varian 1986, Andreoni 1988). While fund-raising mechanisms based on tax rewards and penalties can be designed to solve this problem (e.g., Groves and Ledyard 1977, Walker 1981), they are not available to private organizations with no coercive power, such as charities or civic groups. Contests as incentive mechanisms are different from the above solutions because no power to enforce sanctions is required on the part of the

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institution conducting the tournament. Contests are competitions in which agents spend resources in order to win one or more prizes. The main characteristic is that, independently of success, all participants bear some costs. The present study contributes to two strands of theoretical literature: the papers that examine fund-raising mechanisms on the one hand, and the literature on multiple prize contests on the other.

There exists a large literature that analyzes the use of contests as incentive schemes.[1] A number of recent studies have explored the use of prize-based schemes as fund-raising mechanisms, both theoretically and through experiments. These papers can be divided into two categories. A first one focusing exclusively on lotteries, that is, stochastic contests (e.g., Morgan 2000, Morgan and Sefton 2000, Landry et al. 2006, Lange, List, and Price 2007); and a second group of papers that also consider all-pay auctions, that is, deterministic contests (e.g., Goeree et al. 2005, Orzen 2008, Corazzini, Faravelli, and Stanca 2010, Schram and Onderstal 2009). While Landry et al. (2006) and Lange, List, and Price (2007) analyze multiple prize lotteries, the present study is the first in this literature to examine the effect of multiple prizes in a deterministic contest.[2]

In a seminal paper, Morgan (2000) initiated the literature on prize-based fund-raising mechanisms studying a lottery where one prize is awarded as a way to overcome the free-riding problem. Contributions to the public good entitle to lottery tickets, one ticket is randomly drawn and the holder wins the prize. The public good consists of the revenue of the lottery net of the value of the prize. He considers a model with quasilinear preferences where all players contribute the same amount in equilibrium, independently of their income. Goeree et al. (2005) added an important theoretical contribution, comparing the performance of a winner-pay auction, an all-pay auction and a lottery with a single prize. The paper examines a public good game with a linear production function where agents have heterogeneous preferences, which are private information. Because of the positive externalities associated with the public good, the revenue equivalence does not hold any longer and total expected contributions are shown to be higher with the all-pay auction than with the winner pay. Moreover, the authors prove that the all-pay auction also outperforms the lottery; the intuition is that the lottery is actually an inefficient version of the all-pay auction, that is, the player who contributes the most is the most likely winner, but does not necessarily win.

Several theoretical papers have analyzed multiple prize contests, with different results depending on the specific settings of the model. In a very

[1]Applications have been made to promotions in labor markets (Lazear and Rosen 1981), technological and research races (Wright 1983, Dasgupta 1986, Taylor 1995, Fullerton and McAfee 1999, Windham 1999), credit markets (Broecker 1990), and rent seeking (Tullock 1980) among others.

[2]For simplicity of language, throughout the paper, I will use the term contest meaning a deterministic contest, as opposed to a lottery (i.e., a stochastic contest).

influential study, Moldovanu and Sela (2001) examine a contest with multiple prizes where agents differ in the ability to exert effort, and the ability is private information. They show that when costs are either linear or concave allocating only one prize maximizes the total expected effort exerted by the bidders, while when costs are convex more prizes could be optimal. Lange, List, and Price (2007) consider multiple prize lotteries to finance public goods. In line with Moldovanu and Sela's results, they prove that a single prize is optimal in the case of symmetric risk neutral players, while multiple prizes may maximize contributions when agents are either asymmetric or risk averse. However, the paper on multiple prize contests most closely related to this study is the one by Barut and Kovenock (1998), who study symmetric multiple prize all-pay auctions with complete information. They extend the analysis of first price all-pay auctions with complete information [3] and show that only mixed strategy equilibria exist. Further, expected expenditures are maximized by driving the value of the lowest prize to zero, but are invariant across all configurations leaving the lowest value fixed and the sum of the values constant.

In this paper, I consider a public good game with a linear production function (as in Goeree et al. 2005). I assume that agents have identical valuations, but I allow for heterogenous budget constraints that are private information. The focus of my analysis is on the players' endowments, contrary to most of the previous literature that resolves the modeling trade-off in the opposite direction. Each agent chooses how much of her wealth to allocate to the public good; this money is multiplied by a parameter, which takes a value between one and the number of players, and shared equally among all the agents. The unique Nash equilibrium is to contribute nothing, although it is socially optimal to contribute all the wealth. I overcome this extreme free riding *via* a deterministic contest where several prizes may be awarded. I assume that the social planner has access to a small[4] budget, which can be allocated in form of prizes. The first prize is awarded to the player who contributes the most, the second prize to the player with the second highest contribution, and so on until all prizes are awarded. The social planner determines the prize structure in order to maximize expected total welfare net of the value of the total prize sum.

Differently from the equilibrium described by Morgan (2000), budget heterogeneity and incomplete information enable me to characterize a

[3]The first price all-pay auction with complete information has been utilized extensively in the literature (Dasgupta 1986, Hillman and Samet 1987, Hillman and Riley 1989, Ellingsen 1991, Baye, Kovenock, and De Vries 1993). There exists no pure strategy Nash equilibrium and a complete characterization of its equilibria appears in Baye, Kovenock, and De Vries (1996).

[4]I focus on cases where fund-raisers auction prizes of relatively low value. This seems indeed to be the main source of revenue for most charities. See for example, Goeree et al. (2005), who provide data showing that the vast majority of fund-raisers seek small donations from a large number of donors.

monotone equilibrium in pure strategies, in which the contribution is strictly increasing in the endowment. I prove that it is optimal for the social planner to set the last prize equal to zero, but otherwise total expected contribution is independent of the distribution of the total prize sum among the prizes, thus extending Barut and Kovenock's neutrality result (1998) to the private values budget constrained bidders case. I show that expected total contribution is higher than the value of the total prize sum. There exists a critical level of the budget under which the monotone equilibrium exists independently of the prize structure. For any possible distribution of the endowments, I identify necessary and sufficient conditions for the total prize sum to be below this critical level. Finally, I prove that private provision *via* a contest Pareto-dominates public provision and is higher than total contribution generated by a lottery (as shown by Goeree et al. 2005 for the single prize case).

The paper is structured as follows. In Section 2, I present the model and identify the Nash equilibrium. In Section 3, I solve the designer's problem and I provide necessary and sufficient conditions for the existence of the equilibrium. Section 4 compares private provision *via* a contest with both public provision and private provision in a lottery. Section 5 concludes.

2. The Model

Let us consider n players. Each player i is assumed to have endowment z_i, which is private information. Endowments are drawn independently of each other from the interval $[0, 1]$ according to the distribution function $F(z)$, which is common knowledge, with mean $E[z]$. I assume that $F(z)$ has a continuous and bounded density $F^{'}(z) > 0$. Players play a public good game in which each individual has to choose how much to contribute to the public good. At the same time they take part in a contest in which n prizes are awarded such that $\pi_1 \geq \cdots \geq \pi_{m-1} > \pi_m = \cdots = \pi_n \geq 0$, $1 < m \leq n$ and $\sum_{j=1}^{n} \pi_j = \Pi$. [5] I will call $\pi = (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n$ the vector of prizes. The player with the highest contribution wins π_1, the player with the second highest contribution wins π_2, and so on until all the prizes are allocated. For each player, a strategy $g(z)$ will be the contribution to the public good as a function of the player's endowment and the action space for player i will be the interval $[0, z_i]$. If player i, who has endowment z_i and contributes g_i, wins prize j her payoff is

$$U_i = z_i - g_i + \gamma \frac{\sum_{l=1}^{n} g_l}{n} + \pi_j,$$

where $\gamma \in (1, n)$. Given the range of possible values assumed by γ, the unique Nash equilibrium in the absence of a contest is to contribute

[5]This assumption rules out the trivial case in which n equal prizes are awarded and everyone contributes zero in equilibrium.

nothing, although it is efficient to contribute all the wealth. This is because an individual's opportunity cost of contributing to the public good exceeds the marginal return of investing in it.

Each player i chooses her contribution in order to maximize expected utility (given the other players' contributions and given the values of the different prizes). I will assume that Π is exogenously determined. For a given value of Π, the social planner determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximize the expected value of total welfare net of the value of Π (given the players' equilibrium strategy functions).

2.1. Equilibrium Analysis

I will focus on the case in which the equilibrium strategy $g(z)$ is less than z for any type z on the interval $[0, 1]$. In order to solve the game, it is useful to introduce the function

$$K(z) = \sum_{i=1}^{n} \pi_i \binom{n-1}{i-1} (F(z))^{n-i} (1 - F(z))^{i-1}. \tag{1}$$

Given a vector of prizes π, $K(z)$ is a linear combination of n-order statistics with weights equal to the prizes. If all agents adopt the same strictly increasing strategy $g(z)$, $K(z)$ represents the expected prize of the player with endowment z. The following result will help us identify the equilibrium of the game.

LEMMA 1: *The function $K(z)$ is strictly increasing in z.*

Proof: See Appendix.

Given Lemma (1), at interior solutions for all players (i.e., if budget constraints are non-binding) I am able to characterize a monotone equilibrium, in which the contribution is strictly increasing in the endowment. Later on, I will identify necessary and sufficient conditions for its existence independently of the prize structure.

PROPOSITION 1: *Given a vector π of prizes, at an interior solution for all players the game has a symmetric pure strategy equilibrium given by*

$$g(z) = \frac{n}{n-\gamma} (K(z) - \pi_n).$$

Proof: See Appendix.

It is well known that if players do not have budget constraints, or they all have ample endowments, the all-pay auction has only mixed strategy equilibria. Barut and Kovenock (1998) extend this result to the case of a symmetric multiple prize all-pay auction with complete information, identifying all the equilibria. The introduction of endowments that are private information allows me to construct an equilibrium in pure strategies, which is effectively the purification of the mixed strategy equilibrium described in Barut and Kovenock (1998); the players use their endowments as a randomizing device to coordinate on this equilibrium.[6]

Notice that the equilibrium strategy function defined in Proposition 1 can be rearranged as $g(z) = \frac{1}{1-\frac{\gamma}{n}}(K(z) - \pi_n)$. The latter is the sum of a convergent series with reason $\frac{\gamma}{n}$ and can be expressed as

$$g(z) = (K(z) - \pi_n) \sum_{m=0}^{\infty} \left(\frac{\gamma}{n}\right)^m.$$

The first part of the above equation represents the expected prize, in equilibrium, of a player with endowment z, net of the value of the last prize. This is because a contribution equal to zero would guarantee the agent to win the lowest prize. The multiplier $\sum_{m=0}^{\infty}(\frac{\gamma}{n})^m$ reflects the return to investment in the public good. In standard all-pay auctions, an agent's expected bid equals her expected prize in equilibrium. In our model, if an agent contributes up to her expected prize (net of the last prize) she receives back $\frac{\gamma}{n}$ times the value of her bid because of the public good. This implies that she will add to her contribution this remaining value, from which she will get back an equal proportion, and so on.

If budget constraints are binding for some (or all agents), the equilibrium will be different, but it will nevertheless maintain the same properties, that is, it will be monotone in pure strategies. Figure 1 reports a simple but instructive example in which $F(z) = z$, $n = 3$, $\gamma = \frac{3}{2}$, $\Pi = 1$ and a single prize is awarded. The equilibrium strategy function is $g(z) = \frac{n}{n-\gamma}K(z) = 2z^2$ for $z \in [0, \frac{1}{2})$, while it is equal to z for the interval $[\frac{1}{2}, 1]$.

3. Designer's Problem and Revenue Equivalence

I now consider the maximization problem faced by the designer, assuming that wealth constraints are non-binding for all players. Recall that the social planner determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximize expected total welfare net of the value of Π (given the players' equilibrium strategy functions). In order to analyze the maximization problem, I let the vector of prizes Π be variable, maintaining the assumptions that $\pi_1 \geq \cdots \geq \pi_{m-1} > \pi_m = \cdots = \pi_n \geq 0$, $1 < m \leq n$ and $\sum_{j=1}^{n} \pi_j = \Pi$, and I study the family of functions $K(z, \pi)$. Letting Π variable, at an interior solution for all players, the equilibrium strategy is represented by the following:

[6]Note that this implies that the equilibrium I identified is not unique.

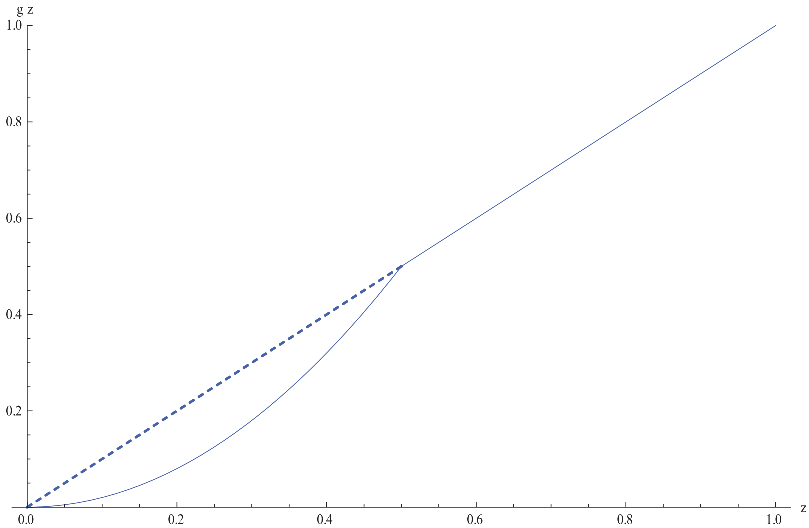

Figure 1: The solid line represents the equilibrium strategy function when $F(z) = z$, $n = 3, \gamma = \frac{3}{2}, \Pi = 1$ and only one prize is assigned. The dashed line is the 45°.

$$\begin{aligned} g(z,\pi) &= \frac{n}{n-\gamma}(K(z,\pi) - \pi_n) \\ &= \frac{n}{n-\gamma}\sum_{i=1}^{n} \pi_i \binom{n-1}{i-1} (F(z))^{n-i}(1-F(z))^{i-1} - \frac{n}{n-\gamma}\pi_n. \end{aligned}$$

The social planner wishes to maximize total expected welfare, which can be expressed as

$$W = nE[z] + (\gamma - 1)n \int_0^1 g(z,\pi)F'(z)\,dz.$$

The first term is a constant representing total expected endowments, while the second term is the total expected provision of public good net of contributions. Recall that in equilibrium $K(z,\pi)$ is the expected prize of a player with endowment z, and therefore $n\int_0^1 K(z,\pi)F'(z)\,dz = \Pi$, independently of the distribution of the total prize sum. This implies that expected total contribution in equilibrium is equal to

$$G(\pi) = n\int_0^1 g(z,\pi)F'(z)\,dz = \frac{n}{n-\gamma}\Pi - \frac{n^2}{n-\gamma}\pi_n.$$

Note that the above expression depends only on the total prize sum and on the value of the last prize. I can then rewrite total expected welfare as

$$W = nE[z] + \frac{n(\gamma - 1)}{n - \gamma}(\Pi - n\pi_n) \tag{2}$$

and I can state the following result.

PROPOSITION 2: *At an interior solution for all players the social planner optimally sets the last prize equal to zero, but otherwise expected total contribution is* $G = \frac{n}{n-\gamma}\Pi$ *independently of the distribution of the total prize sum among the prizes.*

Total expected contribution is higher than the total prize sum. While the standard result in all-pay auctions is the total dissipation of rent (in expectation), in our model overdissipation occurs because of the marginal return of investing in the public good. Furthermore, from expression (2) I can see that in equilibrium total expected welfare net of the value of Π equals $nE[z] + \frac{n(\gamma-1)}{n-\gamma}\Pi$, where $nE[z]$ represents initial welfare.

3.1. Interior Solutions

So far I have assumed that wealth constraints are non-binding for all agents. In order for the revenue equivalence to hold the solution must be interior for all players for any possible distribution of the total prize sum among the prizes. Continuity together with the assumption that $F(z)$ has a bounded density guarantee the existence of the equilibrium, independently of the allocation of prizes, if Π is small enough.

LEMMA 2: *There exists a critical level* $\bar{\Pi}$ *such that the equilibrium strategy function is interior for all players independently of the distribution of the total prize sum across the first* $n-1$ *prizes if and only if* $\Pi \leq \bar{\Pi}$.

I provide necessary and sufficient conditions for Π to be below such a critical value independently of the prize structure, provided that the social planner optimally sets the last prize equal to zero. [7]

Let us define the following object.

DEFINITION 1: *Define the envelope function*

$$V(z) = \max_{\pi}\left\{K(z,\pi)\;\middle|\;\sum_{j=1}^{n}\pi_j = \Pi, \pi_1 \geq \cdots \geq \pi_n, \pi_n = 0\right\}$$

for any z on the interval $[0, 1]$.

If I am able to provide necessary and sufficient conditions for $V(z)$ to be weakly less than z for any z on the interval $[0, 1]$, it will be easy to extend the

[7]Hereafter, unlike the rest of the paper, when writing $K(z, \pi)$ I will refer to $K(z, \pi \mid \sum_{j=1}^{n}\pi_j = \Pi, \pi_1 \geq \cdots \geq \pi_n, \pi_n = 0)$.

result to $g(z, \pi)$. In order to do this, I will define some useful concepts that will help us in the course of our analysis.

DEFINITION 2: *For any i such that $1 \leq i \leq n - 1$:*
(1) define the set $Q^i \subset \mathbb{R}^n$ such that for every $\pi \in Q^i$ it holds that $\pi_1 \geq \cdots \geq \pi_i > \pi_{i+1} = \cdots = \pi_n = 0$ and $\sum_{l=1}^{i} \pi_l = \Pi$.
(2) call $\bar{\pi}^i$ the vector $\pi \in Q^i$ such that $\pi_1 = \cdots = \pi_i = \frac{\Pi}{i}$.

DEFINITION 3: *For any i such that $2 \leq i \leq n - 1$ define the set $\tilde{Q}^i \subset Q^i$ such that for every $\pi \in \tilde{Q}^i$ it holds that $\pi_1 > \pi_i$.*

Obviously $\bar{\pi}^1 \in Q^1$, characterized by ${\pi^1}_1 = \Pi$, ${\pi^1}_l = 0$ for $2 \leq l \leq n$, is the only element of the set Q^1 and $K(z, \bar{\pi}^1) = \Pi(F(z))^{n-1}$.

The next proposition presents necessary and sufficient conditions for $V(z)$ to be weakly less than z on the interval $[0, 1]$.

PROPOSITION 3: *$K(z, \bar{\pi}^i) \leq z$ on the interval $[0, 1]$ for $1 \leq i \leq n - 1$ are necessary and sufficient conditions for $V(z) \leq z$.*

Proof: See Appendix.

Finally, given Proposition 3, by continuity I can establish the following result.

PROPOSITION 4: *Provided that the last prize is equal to zero, $g(z, \pi)$ is interior for any z on the interval $[0, 1]$ independently of the distribution of Π among the first $n - 1$ prizes if and only if $\frac{n}{n-\gamma} K(z, \bar{\pi}^i \mid \sum_{j=1}^{n} \bar{\pi}^i_j = \Pi) \leq z$ on the interval $[0, 1]$ for $1 \leq i \leq n - 1$.*

Proposition 4 tells us that $n - 1$ conditions must be satisfied in order for the equilibrium to exist independently of the distribution of the total prize sum among the first $n - 1$ prizes. The conditions are the following: the equilibrium strategy function must be interior when one prize is awarded, when two *equal* prizes are awarded, when three *equal* prizes are awarded, and so on until $n - 1$ *equal* prizes are assigned. The intuition behind this result can be thought of in the following way. Recall that a player's equilibrium strategy is proportional to her expected prize in equilibrium. Suppose a total prize sum of \$10 and think of the case in which only two prizes are awarded as an example. For any z, the equilibrium strategy will be bounded between the two extreme cases: the case in which \$10 are awarded to the winner and the one where \$5 are awarded to the two highest contributions. For any other allocation (e.g., \$8 to the first and \$2 to the second), $g(z)$ will fall in between these bounds. Whether the single prize is the upper bound or the lower bound will depend on whether it is more likely for z to be the highest or the second highest type.

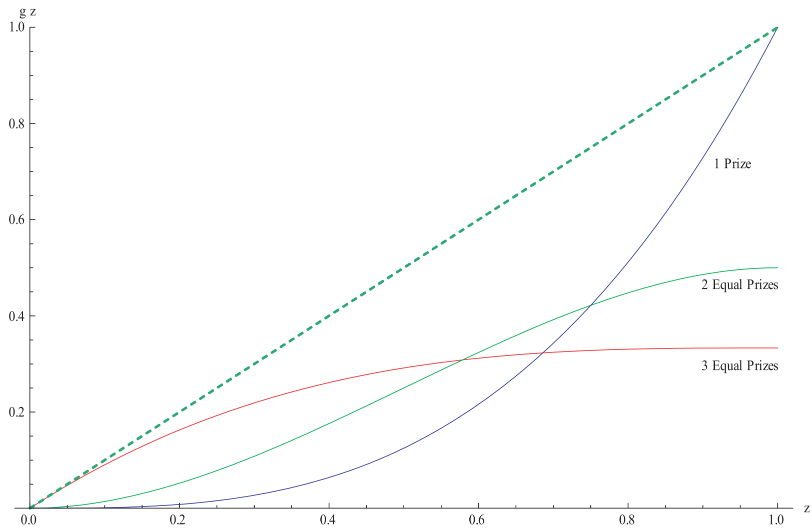

Figure 2: Necessary and sufficient conditions for the case in which $F(z) = z$, $n = 4$, $\gamma = 2$ and $\Pi = 1/2$. I report the equilibrium strategy function for the cases in which one prize, two equal prizes and three equal prizes are awarded. The dashed line represents the 45°.

These conditions are easily satisfied when the endowments are uniformly distributed. Indeed, in this case it is sufficient to check that the equilibrium strategy function is interior when a single prize is assigned. I provide an example in which $n = 4$, $\gamma = 2$, $\Pi = \frac{1}{2}$, and endowments are randomly drawn from a uniform distribution. Figure 2 shows the equilibrium strategy function for the cases in which one prize, two equal prizes and three equal prizes are assigned. As we can see, as long as $\Pi \leq \frac{1}{2}$ all three functions are below the 45° line, guaranteeing that $g(z)$ will be interior for all z independently of the allocation of Π.

4. Contest versus Public Provision and Lottery

I am going to compare the result obtained through a contest with both the welfare generated by public provision and the total contribution resulting from the use of a lottery.

When socially desirable public goods are not privately provided, the obvious alternative is to publicly provide them. Suppose that the social planner has access to a budget equal to $\Pi \leq \bar{\Pi}$. Instead of allocating this sum in form of prizes, he provides an amount of public good equal to Π. I want to analyze how the total expected welfare generated by public provision compares with that resulting form the use of a contest as an incentive scheme.

PROPOSITION 5: *Private provision of public good via a contest, in which the total sum prize* $\Pi \leq \bar{\Pi}$ *is distributed among the* $n-1$ *players who contribute the most, Pareto-dominates public provision. If the social planner uses* $\Pi \leq \bar{\Pi}$ *to publicly provide the public good, the expected total welfare net of the value of* Π *is* $W^P = nE[z] + (\gamma - 1)\Pi$.

Proof: See Appendix.

I now consider the case where the social planner resorts to a lottery to encourage contribution to the public good. When players are symmetric and risk neutral, Lange, List, and Price (2007) show that, for a given prize sum, a single prize lottery raises higher total contributions than a lottery with multiple prizes where each agent is awarded at most one prize. [8] I therefore restrict my attention to a single prize, and start by exploring equilibrium behavior when the budget constraints of all players are non-binding. To do this, let us start by assuming n players whose endowments are drawn independently of each other from the interval $[\underline{z}, \bar{z}]$, with $\underline{z}$ strictly positive, according to the distribution function $F(z)$, which is common knowledge. Assume that the social planner decides to award the sum Π in a lottery with the following properties. If player i with endowment z_i contributes $g_i \in [0, z_i]$, she wins Π with probability $\frac{g_i}{g_i+G_{-i}}$, where G_{-i} is the sum of the contributions of all the other agents. Player i's expected utility is given by

$$E[U(z_i - g_i, \Pi) \mid g_i, G_{-i}] = z_i - g_i + \gamma \frac{G_{-i} + g_i}{n} + \frac{g_i}{g_i + G_{-i}} \Pi.$$

Differentiating with respect to g_i and setting this equal to zero, I obtain the following:

$$\frac{\gamma - n}{n} + \frac{G_{-i}}{(g_i + G_{-i})^2} \Pi = 0.$$

Assuming that total contribution is different from zero[9] and rearranging I obtain player i's best response function, given by the following expression:

$$g_i^* = -G_{-i} + \sqrt[2]{\frac{n}{n-\gamma} \Pi G_{-i}}. \tag{3}$$

[8]On the other hand, it is easy to show that, if agents are risk neutral, a single prize lottery is equivalent to a multiple prize lottery where each player can potentially be awarded all the prizes.

[9]Notice that in equilibrium total contribution will not be zero. In fact, if any other player different from i contributes zero, player i will contribute ε arbitrarily close to zero and win the prize.

Based on Equation (3), I can write an expression for the total contribution when player i plays according to her best response function

$$G(g_i^* \mid G_{-i}) = 2\sqrt{\frac{n}{n-\gamma}\Pi G_{-i}}.$$

Although endowments are private information, notice that z does not enter the first-order condition. Each player will have the same best response function and the contribution in equilibrium will be the same for any z. Therefore, I know that g_i^* will be $\frac{G(g_i^* \mid G_{-i})}{n}$ and I can express it as follows:

$$g_i^* = \frac{2\sqrt{\frac{n}{n-\gamma}\Pi G_{-i}}}{n}. \tag{4}$$

Setting Equations (3) and (4) equal, I obtain an expression for G_{-i} when all players play according to the best response function. This is represented by

$$G_{-i}^* = \frac{(n-1)^2}{n(n-\gamma)}\Pi.$$

Therefore, assuming that wealth constraints are non-binding, in equilibrium all agents will play according to the following:

$$g^L = \frac{n-1}{n(n-\gamma)}\Pi.$$

It is easy to see that $\Pi \leq \frac{n(n-\gamma)}{n-1}\underline{z}$ guarantees that the solution will be interior for all players. Contrary to the equilibrium I identified for the case of a contest, in a lottery all players contribute the same amount (as in Morgan 2000). Total contribution in equilibrium is given by

$$G^L = \frac{n-1}{n-\gamma}\Pi.$$

These results are summarized in the following proposition.

PROPOSITION 6: *Assume n players whose endowments are drawn independently of each other from the interval* $[\underline{z}, \bar{z}]$, *with* $\underline{z}$ *strictly positive, according to the distribution function* $F(z)$, *which is common knowledge. Assume that z is private information. If* $\Pi \leq \frac{n(n-\gamma)}{n-1}\underline{z}$, *the lottery has a symmetric pure strategy equilibrium in which all players contribute* $g^L = \frac{n-1}{n(n-\gamma)}\Pi$ *and total contribution is* $G^L = \frac{n-1}{n-\gamma}\Pi$.

I now extend the analysis to the case in which the players' budget constraints are binding with positive probability, that is, when $\underline{z} = 0$ or $\underline{z} > 0$ but $\Pi > \frac{n(n-\gamma)}{n-1}\underline{z}$. I can prove that total contribution will be less then G^L.

PROPOSITION 7: *If* $\Pi > \frac{n(n-\gamma)}{n-1}\underline{z}$, *with* $\underline{z} \geq 0$, *then total contribution generated by a lottery is strictly lower than* $\frac{n-1}{n-\gamma}\Pi$.

Proof: See Appendix.

Note that in order to prove Proposition 3, I have not resorted to the support of z. The same conditions guarantee the existence of the equilibrium described in Proposition 1 also when endowments are drawn from the interval $[\underline{z}, \bar{z}]$, with $\underline{z} \geq 0$, according to the distribution function $F(z)$, which is common knowledge, with a continuous and bounded density $F'(z) > 0$. In this case, provided that the social planner optimally sets the last prize equal to zero, the expected total contribution generated by a contest is given by

$$G = n \int_{\underline{z}}^{\bar{z}} \frac{n}{n-\gamma} K(z, \pi) F'(z)\, dz = \frac{n}{n-\gamma} \Pi.$$

I can conclude that, when $\Pi \leq \bar{\Pi}$, the expected total contribution raised through a multiple prize deterministic contest is greater than that obtained with a lottery, thus extending the result proved by Goeree et al. (2005) for the single prize case. The intuition is that, being a lottery a stochastic contest (see Tullock 1980), it is not an efficient mechanism: the highest bid is not necessarily awarded the highest prize.

PROPOSITION 8: *If* $\Pi \leq \bar{\Pi}$, *the expected total contribution in a contest, where* Π *is allocated among the* $n-1$ *players who contribute the most, is greater than the total provision generated by a lottery.*

5. Conclusions

Exploring effective ways to fund public goods is a policy question of great importance, given the fundamental role they play in society. There exists an extensive literature on fund-raising mechanisms based on taxes and penalties. However, solutions to the free-riding problem, which do not require coercive power, have only recently started to be studied. In the case of institutions, which are unable to enforce sanctions, such as charities, this difference may be extremely important.

I considered a linear public good game and a deterministic contest. Contrary to the existing literature, the focus of this study is on budget constrained agents. The main characteristics of the model are the possibility of awarding multiple prizes on the one side, and endowment heterogeneity and incomplete information on the other. I assumed that the social planner has access to a small budget and uses it to implement a contest. The first prize is awarded to the player who contributes the most, the second prize to the player with the second highest contribution, and so on until all prizes are awarded. The social planner's objective function is represented by the expected total welfare net of the total prize sum.

I concentrated my analysis on interior solutions and found that there exists a critical level of budget under which wealth constraints are non-binding for all agents, independently of the prize structure. For any possible distribution of wealth, I identified necessary and sufficient conditions for the budget to be below this critical level. Differently from previous papers in this literature, budget heterogeneity and incomplete information about income levels allowed me to characterize a monotone equilibrium, in which the higher the endowment of a player the higher her contribution. I showed that it is optimal for the social planner to set the last prize equal to zero, but otherwise total expected contribution is invariant to all configurations leaving the lowest value fixed. Expected revenue generated through such a mechanism is higher than the total prize sum. Finally, provided interior solutions, I proved that a contest Pareto-dominates public provision of the public good and performs better than a lottery. From a social welfare perspective, whether the highest bidder wins a bigger prize than the second last does not matter at all, inasmuch as the public good provision generated by a unique winner is no more than that resulting from $n-1$ *winners* of equal prizes. In this sense, I can say that winning is not that important after all; what matters is that all players take part in the contest.

Appendix

Proof of Lemma 1. For a given z, $\binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1}$ represents the probability that type z is the ith highest endowment. Consider z_j and z_l such that $0 \leq z_j < z_l \leq 1$. We know that $\binom{n-1}{i-1}(F(z_j))^{n-i}(1-F(z_j))^{i-1} < \binom{n-1}{i-1}(F(z_l))^{n-i}(1-F(z_l))^{i-1}$ for $i=1$ while the opposite is true when $i=n$. Furthermore, notice that $\sum_{i=1}^{n}\binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1} = 1$ for any z on the interval $[0, 1]$. Given the assumption that $\pi_1 \geq \cdots \geq \pi_{m-1} > \pi_m = \cdots = \pi_n \geq 0$, $1 < m \leq n$, $K(z_l)$ assigns higher weights than $K(z_j)$ to higher prizes and lower weights than $K(z_j)$ to lower prizes. Therefore, $K(z_j) < K(z_l)$. ■

Proof of Proposition 1. The expected utility of a player from a choice g can be calculated as

$$E[U(z-g,\pi)\mid g, g_{-i}] = z - g + \gamma\frac{g}{n} + \gamma\frac{G_{-i}}{n} + (\Pr[1\mid g, g_{-i}]\pi_1 + \Pr[2\mid g, g_{-i}]\pi_2 + \cdots + \Pr[n\mid g, g_{-i}]\pi_n),$$

where $\Pr[j \mid g, g_{-i}]$ is the probability of a choice g being jth highest conditional on the other strategies g_{-i}. If all agents adopt the same strictly increasing strategy $g(z)$, then the probability that a candidate with endowment z_i is higher ranked than another randomly chosen candidate is $\Pr[g(z_i) > g(z)] = \Pr[z_i > z] = F(z_i)$. Therefore,

$$(\Pr[1 \mid g, g_{-i}]\pi_1 + \Pr[2 \mid g, g_{-i}]\pi_2 + \cdots + \Pr[n \mid g, g_{-i}]\pi_n) =$$

$$K(z) = \sum_{i=1}^{n} \pi_i \binom{n-1}{i-1} (F(z))^{n-i}(1 - F(z))^{i-1}.$$

Now, given the common strategy $g(z)$, let us suppose that an individual with endowment z chooses $g(\hat{z})$ for some $\hat{z}$, then her expected utility will be

$$z - g(\hat{z}) + \gamma \frac{G_{-i} + g(\hat{z})}{n} + K(\hat{z}),$$

where G_{-i} is the sum of the contributions of all the other players. Differentiating with respect to $\hat{z}$, We obtain

$$\frac{\gamma - n}{n} g'(\hat{z}) + K'(\hat{z}).$$

In equilibrium, the individual with endowment z should choose $g(z)$ so that the above will be equal to zero when $\hat{z} = z$, and we have

$$g'(z) = \frac{n}{n-\gamma} K'(z).$$

A player with the lowest possible endowment $z = 0$ does not contribute to the public good and wins the last prize. This yields the boundary condition $g(0) = 0$. Hence, the solution is

$$g(z) = \frac{n}{n-\gamma}(K(z) - \pi_n).$$

From Lemma (1), we know that the candidate equilibrium function g is strictly monotonic increasing.

Assuming that all players rather than i play according to g, we finally need to show that, for any type z of player i, the contribution $g(z)$ maximizes the expected utility of that type. Let us consider an individual with endowment z. If she plays $g(z) = \frac{n}{n-\gamma}(K(z) - \pi_n)$ her expected utility is given by

$$E[U(z, g(z)) \mid g_{-i}] = z - \frac{\gamma}{n-\gamma} K(z) + \frac{n}{n-\gamma}\pi_n + \frac{\gamma}{n} G.$$

If she deviates and plays $\frac{n}{n-\gamma}(K(\hat{z})-\pi_n)$ for some $\hat{z}\neq z$ her expected utility will be

$$\begin{aligned}E[U(z,g(\hat{z}))\mid g_{-i}] &= z-\frac{n}{n-\gamma}K(\hat{z})+\frac{n}{n-\gamma}\pi_n+\frac{\gamma}{n}\Bigg(G-\frac{n}{n-\gamma}K(z)\\&\quad+\frac{n}{n-\gamma}\pi_n+\frac{n}{n-\gamma}K(\hat{z})-\frac{n}{n-\gamma}\pi_n\Bigg)+K(\hat{z})\\&= z-\frac{\gamma}{n-\gamma}K(z)+\frac{n}{n-\gamma}\pi_n+\frac{\gamma}{n}G.\end{aligned}$$

Therefore, she is indifferent between playing $g(z)$ or any other strategy $\frac{n}{n-\gamma}(K(\hat{z})-\pi_n)$. Since $g(z)$ is strictly increasing in z and $g(0)=0$, it is not restrictive to analyze deviations from the equilibrium strategy as player type deviations. Indeed, if $z\leq g(1)$ this rules out the possibility that she might be better off deviating from $g(z)$ and choosing instead any other contribution less or equal than z. If $z>g(1)$ it is easy to show that she would be worse off playing any strategy greater than $g(1)$. In fact, playing $g(1)$ would already guarantee π_1 and any higher contribution would result in a lower expected utility. ■

Proof of Proposition 3. The necessity of these conditions is obvious. In order to prove sufficiency, it is necessary to present some technical results.

LEMMA 3: *Given a vector $\pi^R\in\mathbb{R}^n$ such that $\sum_{j=1}^{n}\pi_j^R=\Pi$ and $\pi^R{}_1\geq\cdots\geq\pi_n^R$, $\pi_n^R=0$, consider a redistribution of the type $-\Delta\,\pi_i^R=\Delta\,\pi_{i+1}^R$, with $1\leq i\leq n-1$ and $\Delta\,\pi_i^R>0$, and call the resulting vector π^S. Then, $K(z,\pi^S)>K(z,\pi^R)$ for any z such that $F(z)<\frac{n-i}{n}$ and $K(z,\pi^S)<K(z,\pi^R)$ for any z such that $F(z)>\frac{n-i}{n}$.*

Proof: Notice that $\frac{\partial K(z,\pi)}{\partial\pi_i}=\binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1}$. To see how a redistribution of the type $-\Delta\,\pi_i=\Delta\,\pi_{i+1}$ affects $K(z,\pi)$, we study the sign of

$$\begin{aligned}&-\frac{\partial K(z,\pi)}{\partial\pi_i}+\frac{\partial K(z,\pi)}{\partial\pi_{i+1}}\\&\quad=(F(z))^{n-i-1}(1-F(z))^{i-1}\left(-\binom{n-1}{i-1}(F(z))+\binom{n-1}{i}(1-F(z))\right).\end{aligned}\tag{A1}$$

It is the case that expression (A1) >0 for any z such that $F(z)<\frac{\binom{n-1}{i}}{\binom{n-1}{i-1}+\binom{n-1}{i}}$ and (A1) <0 for any z such that $F(z)>\frac{\binom{n-1}{i}}{\binom{n-1}{i-1}+\binom{n-1}{i}}$. Further, it is easy to show that

$$\frac{\binom{n-1}{i}}{\binom{n-1}{i-1}+\binom{n-1}{i}} = \frac{\dfrac{(n-1)!}{i!(n-1-i)!}}{\dfrac{(n-1)!}{(i-1)!(n-i)!}+\dfrac{(n-1)!}{i!(n-1-i)!}} = \frac{n-i}{n}.$$

■

LEMMA 4: *Assume* $1 \leq i \leq n-2$. *Consider a vector* $\pi^B \in \tilde{Q}^{i+1}$. *If* $2 \leq i \leq n-2$ *then* $K(z, \bar{\pi}^{i+1}) > K(z, \pi^B)$ *for any z such that* $F(z) \leq \frac{n-i}{n}$. *If* $i = 1$ *then* $K(z, \bar{\pi}^2) > K(z, \pi^B)$ *for any z such that* $F(z) < \frac{n-1}{n}$ *and* $K(F^{-1}(\frac{n-1}{n}), \pi \mid \pi \in Q^2) = K(F^{-1}(\frac{n-1}{n}), \bar{\pi}^1) = \Pi(\frac{n-1}{n})^{n-1}$.

Proof: Let us first consider the case in which $2 \leq i \leq n-2$. The vector $\bar{\pi}^{i+1}$ can be obtained from vector π^B applying the following algorithm in i steps.

ALGORITHM 1: *Step 1. From vector* π^B *construct vector* π^{B1} *such that* $\pi_1^{B1} = \frac{\Pi}{i+1}, \pi_2^{B1} = \pi_2^B + \pi_1^B - \frac{\Pi}{i+1}, \pi_j^{B1} = \pi_j^B, 3 \leq j \leq i+1$. *Given that* $\pi_2^B \geq \pi_3^B$, *it will now be the case that* $\pi_2^{B1} > \pi_3^{B1} \geq \cdots \geq \pi_{i+1}^{B1}$. *Therefore* $\frac{\Pi}{i+1} + i\pi_2^{B1} > \Pi$. *The last inequality can be rewritten as* $\pi_2^{B1} > \frac{\Pi}{i+1}$; *therefore we can move to the next step and repeat the process.*

Step j, with $2 \leq j \leq i-1$. *From vector* π^{Bj-1} *construct vector* π^{Bj} *such that* $\pi_j^{Bj} = \frac{\Pi}{i+1}, \pi_{j+1}^{Bj} = \pi_{j+1}^{Bj-1} + \pi_j^{Bj-1} - \frac{\Pi}{i+1}, \pi_l^{Bj} = \pi_l^{Bj-1}$ *for* $1 \leq l \leq j-1$ *and* $j+1 \leq l \leq i+1$. *Given that* $\pi_{j+1}^{Bj-1} \geq \pi_{j+2}^{Bj-1}$ *it will now be the case that* $\pi_{j+1}^{Bj} > \pi_{j+2}^{Bj} \geq \cdots \geq \pi_{i+1}^{Bj}$. *Therefore, it is the case that* $j\frac{\Pi}{i+1} + (i+1-j)\pi_{j+1}^{Bj} > \Pi$. *Rearranging the last inequality, we obtain* $\pi_{j+1}^{Bj} > \frac{\Pi}{i+1}$. *This means that we can move to the next step and repeat the process.*

Step i. From vector π^{Bi-1}, *construct vector* π^{Bi} *such that* $\pi_i^{Bi} = \frac{\Pi}{i+1}, \pi_{i+1}^{Bi} = \pi_{i+1}^{Bi-1} + \pi_i^{Bi-1} - \frac{\Pi}{i+1}, \pi_l^{Bi} = \pi_l^{Bi-1}$ *for* $1 \leq l \leq i-1$. *Notice that* $\pi_l^{Bi-1} = \frac{\Pi}{i+1}$ *for* $1 \leq l \leq i-1$. *Therefore* $\pi^{Bi} = \bar{\pi}^{i+1}$.

Notice that from Lemma 3, we know that $K(z, \pi^{Bj}) > K(z, \pi^{Bj-1})$ *for any z such that* $F(z) < \frac{n-j}{n}$ *for* $1 \leq j \leq i$. *Therefore, it must be the case that* $K(z, \bar{\pi}^{i+1}) > K(z, \pi^B)$ *for any z such that* $F(z) \leq \frac{n-i}{n}$.

Consider now the case in which $i = 1$. Notice that $\bar{\pi}_1^2 < \pi_1^B$ and $\bar{\pi}_2^2 > \pi_2^B$. Applying the same algorithm as above from π^B, we obtain $\bar{\pi}_2^2$ after the first step. Applying Lemma 3, we know that $K(z, \bar{\pi}^2) > K(z, \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$. Further, from Lemma 3, we also know that $K(z, \pi \mid \pi \in Q^2) > K(z, \bar{\pi}^1)$ for any z such that $F(z) < \frac{n-1}{n}$ and $K(z, \pi \mid \pi \in Q^2) < K(z, \bar{\pi}^1)$ for any z such that $F(z) > \frac{n-1}{n}$. Therefore, by continuity, we can conclude that $K(F^{-1}(\frac{n-1}{n}), \pi \mid \pi \in Q^2) = K(F^{-1}(\frac{n-1}{n}), \bar{\pi}^1) = \Pi(\frac{n-1}{n})^{n-1}$. ■

LEMMA 5: *Assume* $2 \leq i \leq n-2. K(z, \bar{\pi}^{i+1}) > K(z, \pi \mid \pi \in Q^j)$ *for any* z *such that* $F(z) \leq \frac{n-i}{n}$ *and for* $1 \leq j \leq i$.

Proof: The structure of this proof is in three parts.

First of all, from Lemma 4 we know that $K(z, \bar{\pi}^j) > K(z, \pi \mid \pi \in \tilde{Q}^j)$ for any z such that $F(z) \leq \frac{n-j+1}{n}$ and, given that $2 \leq j \leq i$, for any z such that $F(z) \leq \frac{n-i}{n}$.

For the second part of the proof, let us first assume $j = 1$. Consider a vector $\pi^B \in \tilde{Q}^{i+1}$. We want to show that $K(z, \pi^B) > K(z, \bar{\pi}^1)$ for any z such that $F(z) \leq \frac{n-i}{n}$.

If $2 \leq j \leq i$, consider a vector $\pi^B \in \tilde{Q}^{i+1}$ such that $\pi_l^B = \bar{\pi}_l^j$ for $1 \leq l \leq j-1$. Notice that, obviously, $\pi_j^B < \bar{\pi}_l^j$. We want to show that $K(z, \pi^B) > K(z, \bar{\pi}^j)$ for any z such that $F(z) \leq \frac{n-i}{n}$ if $2 \leq j \leq i-1$ and for any z such that $F(z) < \frac{n-i}{n}$ if $j = i$.

Vector π^B can be obtained from $\bar{\pi}^j$ through the following algorithm in $i+1-j$ steps.

ALGORITHM 2: *Step 1. If* $j = 1$, *from vector* $\bar{\pi}^1$ *construct vector* $\pi^{A1} \in \tilde{Q}^2$ *such that* $\pi_1^{A1} = \pi_1^B$ *and* $\pi_2^{A1} = \Pi - \pi_1^B$. *If* $2 \leq j \leq i$, *from vector* $\bar{\pi}^j$ *construct vector* $\pi^{A1} \in \tilde{Q}^{j+1}$ *such that* $\pi_l^{A1} = \bar{\pi}_l^j = \frac{\Pi}{j}$ *for* $1 \leq l \leq j-1$, $\pi_j^{A1} = \pi_j^B$ *and* $\pi_{j+1}^{A1} = \bar{\pi}_j^j - \pi_j^B = \frac{\Pi}{j} - \pi_j^B$.

Step k, with $2 \leq k \leq i-j$. *From vector* π^{Ak-1}, *construct vector* $\pi^{Ak} \in \tilde{Q}^{j+k}$ *such that* $\pi_l^{Ak} = \pi_l^{Ak-1}$ *for* $1 \leq l \leq j = k-2$, $\pi_{j+k-1}^{Ak} = \pi_{j+k-1}^B$, *and* $\pi_{j+k}^{Ak} = \pi_{j+k-1}^{Ak-1} - \pi_{j+k-1}^B$.

Step $i+1-j$. *From vector* π^{Ai-j}, *construct vector* $\pi^{Ai+1-j} \in \tilde{Q}^{i+1}$ *such that* $\pi_l^{Ai+1-j} = \pi_l^{Ai-j}$ *for* $1 \leq l \leq i-2$, $\pi_i^{Ai+1-j} = \pi_i^B$, *and* $\pi_{i+1}^{Ai+1-j} = \pi_i^{Ai-j} - \pi_i^B$. *Notice that* $\pi_{i+1}^{Ai+1-j} = \pi_{i+1}^B$ *and* $\pi^{Ai+1-j} = \pi^B$ *by construction.*

From Lemma 3, we know that $K(z, \pi^{Ak}) > K(z, \pi^{Ak-1})$ *for any* z *such that* $F(z) < \frac{n+1-k}{n}$. *Therefore, if* $1 \leq j \leq i-1$ *then* $K(z, \pi^B) > K(z, \bar{\pi}^j)$ *for any* z *such that* $F(z) \leq \frac{n-i}{n}$. *If* $j = i$ *then* $K(z, \pi^B) > K(z, \bar{\pi}^j)$ *for any* z *such that* $F(z) < \frac{n-i}{n}$ *and* $K(F^{-1}(\frac{n-i}{n}), \pi^B) = K(F^{-1}(\frac{n-i}{n}), \bar{\pi}^j)$.

Finally, from Lemma 4 we know that $K(z, \bar{\pi}^{i+1}) > K(z, \pi^B)$ for any z such that $F(z) \leq \frac{n-i}{n}$. Therefore, $K(z, \bar{\pi}^{i+1}) > K(z, \pi \mid \pi \in \tilde{Q}^j)$ for any z such that $F(z) \leq \frac{n-1}{n}$. ■

LEMMA 6: *Assume* $2 \leq i \leq n-2$. *Consider a vector* $\pi^B \in \tilde{Q}^{i+1}$ *such that* $\pi_1^B > \pi_j^B$, *with* $2 \leq j \leq i$. *Assume a vector* $\pi^C \in \tilde{Q}^{i+1}$ *such that* $\pi_l^C = \pi_l^B$ *for* $j+1 \leq l \leq i+1$ *and* $\pi_1^C = \cdots = \pi_j^C = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$. *If* $3 \leq j \leq n-2$ *then* $K(z, \pi^C) > K(z, \pi^B)$ *for any* z *such that* $F(z) \leq \frac{n-j+1}{n}$. *If* $j = 2$ *then* $K(z, \pi^C) > K(z, \pi^B)$ *for any* z *such that* $F(z) < \frac{n-1}{n}$ *and* $K(F^{-1}(\frac{n-1}{n}), \pi^C) > K(F^{-1}(\frac{n-1}{n}), \pi^B)$.

Proof: Notice that $\pi_1^B > \pi_1^C$ and $\pi_j^B < \pi_j^C$. Vector π^C can be obtained from vector π^B applying the following algorithm in $j-1$ steps.

ALGORITHM 3: *Step 1. From vector π^B construct vector π^{B1} such that $\pi_1^{B1} = \pi_1^C$, $\pi_2^{B1} = \pi_2^B + \pi_1^B - \pi_1^C$, $\pi_l^{B1} = \pi_l^B$ for $3 \le l \le i+1$. Given that $\pi_2^B \ge \pi_3^B$ it will now be the case that $\pi_2^{B1} > \pi_3^{B1} \ge \cdots \ge \pi_{i+1}^{B1}$. Therefore, $\pi_1^{B1} + (j-1)\pi_2^{B1} > \Pi - \sum_{l=j+1}^{i+1} \pi_l^B$. Since $\pi_1^{B1} = \pi_1^C = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$, the last inequality can be rearranged as $\pi_2^{B1} > \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$. Therefore, we can move to the next step and repeat the process.*

Step k, $2 \le k \le j-2$. From vector π^{Bk-1}, construct vector π^{Bk} such that $\pi_k^{Bk} = \pi_k^C$, $\pi_{k+1}^{Bk} = \pi_{k+1}^{Bk-1} + \pi_k^{Bk-1} - \pi_k^C$, $\pi_l^{Bk} = \pi_l^{Bk-1}$ for $1 \le l \le k-1$ and $k+2 \le l \le i+1$. Notice that, by construction $\pi_l^{Bk} = \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$ for $1 \le l \le k$ and $\pi_l^{Bk} = \pi_l^B$ for $k+2 \le l \le i+1$. Given that $\pi_{k+1}^{Bk-1} \ge \pi_{k+2}^{Bk-1}$, it will now be the case that $\pi_{k+1}^{Bk} > \pi_{k+1}^{Bk} \ge \cdots \ge \pi_{i+1}^{Bk}$. Therefore $\frac{k}{j}(\Pi - \sum_{l=j+1}^{i+1} \pi_l^B) + (j-k)\pi_{k+1}^{Bk} > \Pi - \sum_{l=j+1}^{i+1} \pi_l^B$. The last inequality can be rearranged as $\pi_{k+1}^{Bk} > \frac{\Pi - \sum_{l=j+1}^{i+1} \pi_l^B}{j}$. Therefore, we can move to the next step and repeat the process.

Step $j-1$. From vector π^{Bj-2} construct vector π^{Bj-1} such that $\pi_{j-1}^{Bj-1} = \pi_{j-1}^C$, $\pi_j^{Bj-1} = \pi_j^{Bj-2} + \pi_{j-1}^{Bj-2} - \pi_{j-1}^C$, $\pi_l^{Bj-1} = \pi_l^{Bj-2}$ for $1 \le l \le j-2$ and $j+1 \le l \le i+1$. Notice that $\pi^{Bj-1} = \pi^C$ by construction.

From Lemma 3, we know that $K(z, \pi^{Bk}) > K(z, \pi^{Bk-1})$ for any z such that $F(z) < \frac{n-k}{n}$ for $1 \le k \le j-1$.

This means that if $3 \le i \le n-3$ then, by construction, we will have $K(z, \pi^C) > K(z, \pi^B)$ for any z such that $F(z) \le \frac{n-j+1}{n}$. If $i = 2$ then j will necessarily be equal to 3 and, by construction, we will have $K(z, \pi^C) > K(z, \pi^B)$ for any z such that $F(z) < \frac{n-1}{n}$. Further it will be the case that $K(F^{-1}(\frac{n-1}{n}), \pi^C) > K(F^{-1}(\frac{n-1}{n}), \pi^B)$. ■

LEMMA 7: *Consider a vector $\pi^C \in \tilde{Q}^{i+1}$ such that $\pi_{i+1}^C = x$, $\pi_j^C = \frac{\Pi - x}{i}$ with $0 < x < \frac{\Pi}{i+1}$ for $1 \le j \le i$ and $2 \le i \le n-2$. If $K(z, \pi^C) > K(z, \bar{\pi}^{i+1})$ then $K(z, \bar{\pi}^i) > K(z, \pi^C)$.*

Proof: The inequality $K(z, \pi^C) > K(z, \bar{\pi}^{i+1})$ can be rewritten as

$$\frac{\Pi - x}{i}\left((F(z))^{n-1} + \cdots + \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1}\right)$$
$$+ x\binom{n-i}{i}(F(z))^{n-i-1}(1-F(z))^i$$
$$- \frac{\Pi}{i+1}((F(z))^{n-1} + \cdots + (F(z))^{n-i-1}(1-F(z))^i) > 0.$$

The previous expression can be rearranged as

$$\left(\frac{\Pi - x}{i} - \frac{\Pi}{i+1}\right)\left((F(z))^{n-1} + \cdots + \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1}\right)$$
$$> \left(\frac{\Pi}{i+1} - x\right)(F(z))^{n-i-1}(1-F(z))^{i}. \tag{A2}$$

Call A the expression $((F(z))^{n-1} + \cdots + \binom{n-1}{i-1}(F(z))^{n-i}(1-F(z))^{i-1})$, and call B the expression $(F(z))^{n-i-1}$ $(1-F(z))^{i}$. Inequality (A2) is satisfied for $\frac{A}{B} > i$.

The inequality $K(z, \bar{\pi}^{i}) > K(z, \pi^{C})$ can be rewritten as

$$\frac{\Pi}{i}A - \frac{\Pi - x}{i}A - xB > 0. \tag{A3}$$

Inequality (A3) is satisfied for $\frac{A}{B} > i$. ■

From Lemma (5), we know that $K(z, \bar{\pi}^{i+1}) > K(z, \pi \mid \pi \in Q^{j})$ for any z such that $F(z) \leq \frac{n-i}{n}$ and for $2 \leq i \leq n-2$ and $1 \leq j \leq i$. In particular, this means that $V(z)$ will be equal to $K(z, \bar{\pi}^{n-1})$ for any z such that $0 \leq F(z) \leq \frac{2}{n}$. For those z such that $\frac{2}{n} \leq F(z) \leq \frac{3}{n}$ we must check the family of functions $K(z, \pi \mid \pi \in Q^{n-1})$ and $K(z, \bar{\pi}^{n-2})$. In general, assuming $0 \leq i \leq n-3$, in order to find $V(z)$ for those z such that $\frac{n-i-1}{n} \leq F(z) \leq \frac{n-i}{n}$, we must check the families of functions $K(z, \pi \mid \pi \in Q^{j})$ for $i+2 \leq j \leq n-1$ and the function $K(z, \bar{\pi}^{i+1})$.

Consider now a vector $\pi^{C} \in Q^{i+1}$ such that $\pi^{C}_{1} = \cdots = \pi^{C}_{i} > \pi^{C}_{i+1}$, for $2 \leq i \leq n-2$. From Lemma (6) I know that, for those z such that $\frac{n-i}{n} < F(z) \leq \frac{n-i+1}{n}$, the function $K(z, \pi^{C})$ is greater than any other function of the family $K(z, \pi \mid \pi \in Q^{i+1})$ with the exclusion of $K(z, \bar{\pi}^{i+1})$.

From Lemma (7) though, we know that if $K(z, \pi^{C}) > K(z, \bar{\pi}^{i+1})$ then it is the case that $K(z, \bar{\pi}^{i}) > K(z, \pi^{C})$.

Therefore, in order to find the envelope function $V(z)$ for those z such that $\frac{2}{n} \leq F(z) \leq \frac{3}{n}$, it will be sufficient to check the two functions $K(z, \bar{\pi}^{n-1})$ and $K(z, \bar{\pi}^{n-2})$. In general, assuming $0 \leq i \leq n-3$, in order to find $V(z)$ for those z such that $\frac{n-i-1}{n} \leq F(z) \leq \frac{n-i}{n}$ we must check the functions $K(z, \bar{\pi}^{j})$ for $i+1 \leq j \leq n-1$.

From this follows that $K(z, \bar{\pi}^{i}) \leq z$ for $1 \leq i \leq n-1$ are sufficient conditions for $V(z) \leq z$ on the interval $[0, 1]$. ■

Proof of Proposition 5. If the social planner uses $\Pi \leq \bar{\Pi}$ to provide the public good the expected total welfare net of the value of Π is given by

$$W^{P} = n\int_{0}^{1}\left(z + \frac{\gamma}{n}\Pi\right)F'(z)\,dz - \Pi$$
$$= nE[z] + (\gamma - 1)\Pi. \tag{A4}$$

From Equation (2) we know that, if the last prize is equal to zero, the expected total welfare generated by a contest is equal to

$$W = nE[z] + \frac{n(\gamma - 1)}{n - \gamma}\Pi,$$

which is strictly greater than expression (A4). ∎

Proof of Proposition 7. It is easy to see that total contribution will necessarily be less than $\frac{n-1}{n-\gamma}\Pi$ if $\Pi > \frac{n(n-\gamma)}{n-1}\bar{z}$. The case in which $\frac{n(n-\gamma)}{n-1}\underline{z} < \Pi < \frac{n(n-\gamma)}{n-1}\bar{z}$ is more interesting. Suppose that total expected contribution in equilibrium was greater or equal than $\frac{n-1}{n-\gamma}\Pi$. This would imply $G^*_{-i} \geq \frac{(n-1)^2}{n(n-\gamma)}\Pi$ in expectation. Moreover, since a positive mass of agents will not afford to contribute $g^L = \frac{n-1}{n(n-\gamma)}\Pi$, there should be a nonzero mass of players whose individual contributions exceed g^L. However, this is not possible. From Equation (3), we can see that $g^*\ (G_{-i})$ is a concave function with global maximum in $G_{-i} = \frac{n}{n-\gamma}\frac{\Pi}{4}$. This maximum coincides with $\frac{(n-1)^2}{n(n-\gamma)}\Pi$ when $n = 2$, while it is strictly less for $n > 2$. This implies that, if G^*_{-i} was greater or equal than $\frac{(n-1)^2}{n(n-\gamma)}\Pi$, it could not be optimal for player i to contribute more than $g^L = \frac{n-1}{n(n-\gamma)}\Pi$. ∎

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