(Don't) Make My Vote Count

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Abstract
Proponents of proportional electoral rules often argue that majority rule depresses turnout and may lower welfare due to the "tyranny of the majority" problem. The present paper studies the impact of electoral rules on turnout and social welfare. We analyze a model of instrumental voting where citizens have private information over their individual cost of voting and over the alternative they prefer. The electoral rule used to select the winning alternative is a combination of majority rule and proportional rule. Results show that the above arguments against majority rule do not hold in this set up. Social welfare and turnout increase with the weight that the electoral rule gives to majority rule in a close election, while they are independent of the electoral rule when the expected size of the minority tends to zero. However, more proportional rules can increase turnout within the minority group; this effect is stronger the smaller the minority group. We provide a general version of the competition effect, i.e. that turnout in close elections is higher than in biased elections, independently of the systems adopted in the two cases.

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Because the majority ought to prevail over the minority, must the majority have all of the votes, the minority none? Is it necessary that the minority should not even be heard?

John Stuart Mill (1861), *Considerations on Representative Government*.

1. Introduction

In May 2011, the UK held a referendum on its electoral rule. The Liberal Democrat party and the civic initiative "Make My Vote Count" campaigned in favor of abandoning the first-past-the-post (FPTP) system, with the long-term goal of shifting to a proportional representation (PR) system. Although the reform was rejected, that referendum became part of a recent movement in favor of PR, which has succeeded so far in New Zealand and several Canadian states. As a matter of fact, most western democracies shifted to PR systems during the first wave of this movement, spanning over the last third of the 19th century and the first third of the 20th, up to the downfall of the Weimar Republic (Blais et al., 2005).

There were two main arguments against FPTP put forward by the "Make My Vote Count" movement. The first relates to the *wasted votes* phenomenon, namely that majority rule can potentially dismiss the votes of a large part of the electorate.[1] The wasted votes phenomenon is particularly worrisome in multi-party systems and was especially acute in the UK 2005 national election, when more than half of the votes were cast in favor of losing candidates. This concern relates also to the *tyranny of the majority* problem (Madison et al., 1788; Mill, 1861), that is, the fact that under majority rule just above half of the electorate might impose its preferred alternative on the other half.[2] A second modern criticism of FPTP is that it discourages supporters of minoritarian alternatives from voting, as their favorite choice is much less likely to win. That in turn depresses turnout, increases political disaffection and weakens democratic legitimacy. Because PR is not a winner-take-all electoral rule and makes all votes "count", i.e. have an impact on the outcome, supporters of proportional voting rules propose it as a remedy to the "tyranny of the majority" problem and the low turnout associated with majority rule.

This paper studies how electoral rules affect overall turnout, social welfare and the representation of minority groups. We present a two-party model of costly instrumental voting. Citizens have private information over their individual cost of voting and over the alternative they prefer. The electoral rule used to select the winning alternative is a combination of majority and proportional rule.[3] Our results suggest that the arguments against majority rule outlined above are wrong. We show that in *tight elections*, that is, when citizens' probabilities of supporting each alternative are close enough, turnout increases with the weight the electoral rule gives to majority rule. The same applies to social welfare: it is the highest under pure majority rule. On the other hand, we show that in *biased elections*, that is, when the expected size of the minority group tends to zero, proportional rule is not welfare enhancing either, because social welfare is independent of the electoral rule that is adopted.

The driving force behind these seemingly counterintuitive results is the fundamental trade off that proportional and majority rule offer between the *frequency* and the *magnitude* of a vote's impact. Under majority rule, a voter can only have an impact on the outcome if she is pivotal, i.e. if her vote breaks or creates a tie. Although relatively infrequent, the magnitude of this impact is big. On the other hand, proportional rule ensures that votes in favor of an alternative always increase its winning probability. However, the magnitude of that impact is relatively small. As it turns out, when the minority group is expected to be large, magnitude is a much stronger force than frequency, resulting in the majority providing stronger incentives to vote than proportional rule. The difference between the strengths of these two effects vanishes as citizens' probability of supporting the minority alternative tends to zero. In the limit, when the minority group has zero size, a vote has positive impact only if everybody else abstains. In turn, that impact is the same regardless of the electoral rule employed. Hence, in that case, social welfare is the same under any electoral rule.

Finally, we compare overall and minority turnout for different electoral rules and different types of elections. We show that overall turnout is *always* higher in tight elections than in biased ones, independently of the electoral rule. We refer to this as the *generalized competition effect*. Regarding the participation of minority groups, we show that the "Make My Vote Count" initiative was right for the case of biased elections, where more proportional rules foster turnout within the minority group. Moreover, minority supporters vote more often under proportional rule when the election is biased than when it is tight. This *reversed competition effect* is due to the large magnitude that the vote of a minority supporter has in biased elections. In addition, the collective action problem faced by the majority group in this type of elections is so severe that, for any electoral rule, minority supporters vote more often than majority supporters (*underdog effect*).

The present work contributes to the literature on costly voting under majority rule, which became prominent with the seminal papers by Ledyard (1981, 1984) and Palfrey and Rosenthal (1983, 1985). In another key contribution, Börgers (2004) performs a welfare comparison of majority voting, random decision making and compulsory voting for the case in which the electorate is evenly split in expectation. Krasa and Polborn (2009) generalize Börgers' model to the case where citizens are not equally likely to support each alternative, focusing on the welfare properties of mandatory voting. In a recent working paper, Bognar et al. (2012) show that, given certain conditions, welfare is maximized by a sequential voting procedure. Also related to our analysis, Goeree and Grosser (2007) and Lockwood and Ghosal (2009) consider models where citizens' preferences are correlated. Finally, a number of papers study the properties of costly voting models in large elections (see, e.g., Campbell, 1999; Myerson, 1998, 2000; Taylor and Yildirim, 2010). Krishna and Morgan (2012) show that in large elections majority rule is the welfare maximizing rule in the limit, in that it picks the "correct" alternative, while the cost of voting converges to zero as well as turnout. When the electorate grows without bound, turnout converges to zero independently of the electoral rule (see Faravelli et al., 2013). Different ways to generate positive limiting turnout have

been proposed, such as mobilization (Morton, 1991; Shachar and Nalebuff, 1999), ethical voting (Coate and Conlin, 2004; Feddersen and Sandroni, 2006), aggregate uncertainty on the popularity of the majoritatian alternative (Myatt, 2012) or other-regarding preferences (Evren, 2012; Faravelli et al., 2013). However, we chose to abstract away from this issue and focus on finite elections, in order to analyze the most basic driving forces of voting decisions without adding any potentially confounding factors.

Our study also belongs to the long-lasting debate on the relative merits of proportional and majority rule. This debate traces back to the classical works of John Stuart Mill (1861) and Thomas Hare (1865), which spread the belief in PR as the "most democratic system" (Blais et al., 2005). Two recent studies have compared the two systems in a costly voting framework. Herrera et al. (2014) study turnout and power sharing with population uncertainty and provide experimental evidence backing their results. They show that overall turnout is higher under majority rule than under PR for tight elections but that this result reverses for biased elections. Moreover, the underdog effect is present under both rules but it is stronger under PR. In a concurrent paper, Kartal (2014) studies turnout and welfare properties of different electoral rules for the case of both homogeneous and heterogenous costs, the latter being comparable to our framework. She shows that welfare is maximized by majority rule in close elections, although the welfare difference between proportional and majority rule disappears as the population grows because of the diminishing turnout. Both Herrera et al. (2014) and Kartal (2014) prove the underdog effect for any ex ante party size, but cannot say much about intermediate electorate systems. While we only focus on tight and landslide elections, our paper complements the existing literature by providing general results on the electoral system, by considering combinations of the majoritarian and proportional rule. In so doing, we confirm a general version of the competition effect which the related literature could not prove, i.e. that turnout in close elections is higher than in biased elections, even when two different electoral systems are adopted.

The remainder of the paper is organized as follows. In the next section we describe our set up and prove the existence of a type-symmetric equilibrium. We then analyze equilibrium uniqueness, turnout and social welfare in tight elections. Section 4 performs the same analysis for the case of biased elections. Section 5 compares turnout in these two polar cases and Section 6 concludes. All proofs and technical or auxiliary lemmas are in the Appendix.

2. The model

We consider a model of costly voting with two possible alternatives, A and B, and n + 1 risk neutral citizens. Each citizen has got the same ex ante independent probability $\alpha \geq \frac{1}{2}$ of being a supporter of A and $1-\alpha$ of supporting B. While α is common knowledge, a citizen's preference is private information. We will refer to A as the majoritarian alternative and to its supporters as the majority group; and to B as the minoritarian alternative and to its supporters as the minority group.

The benefit for an individual is equal to 1 if her preferred alternative is chosen and 0 otherwise. Citizens decide simultaneously whether to vote or abstain. If a citizen participates in the election she bears a cost. We call c_i individual i's cost of voting. Each voter's cost is randomly and independently drawn from the differentiable distribution $F(c)$ on the support $[\underline{c}, \bar{c}]$, with $0 < \underline{c} < \frac{1}{2}$. We assume that $F'(c) = f(c)$ is strictly positive on the whole support $[\underline{c}, \bar{c}]$. While an individual's cost is private information, $F(c)$ is publicly known.

A voting rule aggregates votes to determine the outcome of the election. Denote by j the number of votes for alternative $z = A$, B and l the number of votes for the other alternative. According to *majority rule* the probability that z is chosen is

$$q_M(j,l) = \begin{cases} 1 & \text{if } j > l, \\ \frac{1}{2} & \text{if } j = l, \\ 0 & \text{if } j < l, \end{cases}$$

whereas under *proportional rule* the probability of z being chosen is

$$q_P(j,l) = \begin{cases} \dfrac{j}{j+l} & \text{if } j, l \neq 0, \\ \dfrac{1}{2} & \text{if } j, l = 0. \end{cases}$$

Since voters are risk neutral, the two voting rules described above can be interpreted as two different allocation rules in a contest between two parties for a prize of unitary value. On the one hand, majority rule is a standard winner-takes-all contest where the prize is entirely allocated to the alternative that receives the most votes. On the other hand, proportional rule assigns to each alternative a share of the prize proportional to its share of the votes. This, in turn, is the typical stylized representation of PR based on the idea that parliamentary seats are assigned to parties proportionally to their vote shares. Under either voting rule, each individual receives a benefit equal to the share of the prize assigned to her preferred alternative.

We consider the family of voting rules Ω_λ that includes all convex combinations of majority and proportional rule. These electoral rules represent systems where choices, like parliamentary seats, are made partly by PR and partly by FPTP within districts.[4] More formally, we define a voting rule as a function $q_\lambda : \mathbb{N} \times \mathbb{N} \to [0, 1]$, where q_λ represents the probability that z is chosen and is defined as

$$q_\lambda(j,l) = \lambda q_M + (1 - \lambda) q_P$$

The value of λ determines the weight assigned to majority rule. Note that $\lambda = 1$ describes pure majority rule and $\lambda = 0$ refers to pure proportional rule.[5] Our voting game can thus be summarized by the collection $\Lambda = \{n, \alpha, \lambda, F(c)\}$.

The solution concept that we employ is Bayesian–Nash equilibrium (BNE). As is customary in the literature, we restrict our attention to type-symmetric BNE, that is, those in which all citizens supporting the same alternative follow the same

strategy. Participation decisions depend on the realization of the individual voting cost. Formally, a strategy is a mapping $s_z : [\underline{c}, \bar{c}] \to \{0, 1\}$, where a choice of 0 denotes abstaining and 1 denotes voting. A strategy profile $\{s_A, s_B\}$ is a BNE of the game Λ if for each alternative $z \in \{A, B\}$ and each individual supporting alternative z, $s_z(c_i)$ maximizes an individual's expected payoff given that all other individuals adhere to s_z. Note that voting for the least preferred alternative is strictly dominated by abstaining. Consequently, in equilibrium, citizens will either abstain or vote for their preferred alternative. It is possible to characterize citizens' strategies through cut-off values $\hat{c}_A$ and $\hat{c}_B$, such that

$$s_z(c_i) = \begin{cases} 1 & \text{if } c_i \leq \hat{c}_z, \\ 0 & \text{if } c_i > \hat{c}_z. \end{cases} \tag{1}$$

Before proving the existence of a BNE, we introduce two objects that will be useful for our analysis. Suppose that the strategy defined by (1) and the thresholds $\hat{c}_A$ and $\hat{c}_B$ constitute an equilibrium. First, the expected benefit from abstaining for a supporter of A is given by

$$U_A(\hat{c}_A, \hat{c}_B) = \sum_{v=0}^{n} \sum_{v_A=0}^{v} \binom{n}{v} \binom{v}{v_A} [\alpha F(\hat{c}_A)]^{v_A} [(1-\alpha) F(\hat{c}_B)]^{v-v_A} \\ \times [1 - \alpha F(\hat{c}_A) - (1-\alpha) F(\hat{c}_B)]^{n-v} q_\lambda(v_A, v - v_A), \tag{2}$$

where v is the number of citizens who turn out to vote, and v_A is the number of votes cast for alternative A. This expression adds up the benefit from abstaining, represented by $q_\lambda(v_A, v - v_A)$, across all possible realizations of v and v_A, given that the ex ante probability of an individual voting for A is $\alpha F(\hat{c}_A)$, while she votes for B with ex ante probability $(1 - \alpha) F(\hat{c}_B)$.

Second, the net expected benefit from voting for a supporter of A is equal to

$$B_A(\hat{c}_A, \hat{c}_B) = \sum_{v=0}^{n} \sum_{v_A=0}^{v} \binom{n}{v} \binom{v}{v_A} [\alpha F(\hat{c}_A)]^{v_A} [(1-\alpha) F(\hat{c}_B)]^{v-v_A} \\ \times [1 - \alpha F(\hat{c}_A) - (1-\alpha) F(\hat{c}_B)]^{n-v} \\ \times [q_\lambda(v_A + 1, v - v_A) - q_\lambda(v_A, v - v_A)]. \tag{3}$$

Similarly, this expression adds up the net benefit from going to vote (the last term in square brackets) across all possible realizations of v and v_A. In an analogous way, it is possible to calculate $U_B(\hat{c}_A, \hat{c}_B)$ and $B_B(\hat{c}_A, \hat{c}_B)$, representing the expected benefit from abstaining and the net expected benefit from voting for a supporter of B, respectively.

The following proposition characterizes existence of the type-symmetric BNE.

Proposition 1. *There exists a pure strategy type-symmetric BNE of the game* $\Lambda = \{n, \alpha, \lambda, F(c)\}$ *characterized by the voting strategy in* (1) *and thresholds* c_A^* *and* c_B^*.

Before proceeding, let us define three concepts which will be the object of our analysis. First, we will call $F(c_z^*)$ the *individual turnout probability* within the group of supporters of z. The second concept is *ex ante voter turnout*, defined as

$$\alpha F(c_A^*) + (1 - \alpha)F(c_B^*),$$

which represents a citizen's ex ante probability of voting before learning her type. Finally, *ex ante social welfare* is given by

$$W(c_A^*, c_B^*) =$$

$$\alpha \left[\int_{\underline{c}}^{c_A^*} (U_A(c_A^*, c_B^*) + B_A(c_A^*, c_B^*) - c)f(c)\, dc + \int_{c_A^*}^{\bar{c}} U_A(c_A^*, c_B^*)f(c)\, dc \right]$$

$$+ (1 - \alpha) \left[\int_{\underline{c}}^{c_B^*} (U_B(c_A^*, \hat{c}_B) + B_B(c_A^*, c_B^*) - c)f(c)\, dc + \int_{c_B^*}^{\bar{c}} U_B(c_A^*, c_B^*)f(c)\, dc \right].$$

$$(4)$$

This expression represents the welfare of an individual before learning her type and it is given by the weighted average of the expected utility of an A-supporter and that of a supporter of B. In turn, each of these two is represented by the sum of the utility of an abstainer and that of a voter. As we know, in equilibrium, supporters of alternative z with a cost higher than c_z^* abstain and receive benefit $U_z(c_A^*, c_B^*)$. Citizens with a lower cost vote in equilibrium, and their utility is found by adding the benefit from abstaining, $U_z(c_A^*, c_B^*)$, and the net benefit from voting, $B_z(c_A^*, c_B^*)$, and subtracting the cost c.

Note that all of the three objects defined above are continuous in c_A^* and c_B^*. Throughout the paper, to simplify the language, we refer to ex ante voter turnout and ex ante social welfare as turnout and social welfare, respectively.

In the following sections, we will analyze the properties of these three objects as a function of the electoral rule q_λ and the expected size of the two groups of supporters. We begin by exploring two polar cases in detail. The case where citizens' probabilities of supporting each alternative are similar, i.e. $\alpha \approx \frac{1}{2}$, and the case where their probability of supporting the minoritarian alternative B tends to zero, i.e. $\alpha \to 1$. We refer to these cases as *tight elections* and *biased elections*, respectively. For each case, we characterize equilibrium uniqueness, and analyze turnout and social welfare as a function of the electoral rule q_λ.

3. Tight elections

In this section, we analyze the case where citizens' probabilities of supporting each alternative are relatively close. On the one hand, this is the case where the argument in favor of PR as a potential *welfare* improving system seems to be the strongest possible. Indeed, the "tyranny of the majority" problem associated with a winner-takes-all election is expected to be most acute, since, in expectation, a large share of the population will see their favorite alternative defeated. On the other hand,

though, the "Make My Vote Count" argument that PR fosters *turnout* among minority supporters would seem rather weak in a close election where the minority group is expected to be large. We explore these opposed arguments first by studying the knife-edge case $\alpha = \frac{1}{2}$, where the electorate is expected to be equally split between A and B, and then by extending the analysis to a neighborhood of that value.

When each citizen is equally likely to support either of the two alternatives it is possible to characterize the equilibrium by using just a single strategy $s : [\underline{c}, \bar{c}] \rightarrow \{0, 1\}$. Following the same logic as above, a strategy can be now fully described by a single cut-off cost $\hat{c}$ such that citizens vote for their preferred alternative if and only if $c_i \leq \hat{c}$. Hence, any equilibria we characterize at $\alpha = \frac{1}{2}$ will be (fully) symmetric.

We begin by showing that at $\alpha = \frac{1}{2}$ the game has a unique symmetric BNE.

Lemma 1. *When the electorate is evenly split in expectation, i.e.* $\alpha = \frac{1}{2}$, *there exists a unique symmetric BNE such that* $c_A^* \epsilon (\underline{c}, \bar{c}]$.

Note that for the case of majority rule (i.e. $\lambda = 1$) uniqueness was already proven by Börgers (2004). The previous lemma generalizes that result to the use of any electoral rule q_λ.

Having established equilibrium uniqueness, we can now safely perform comparative statics on turnout and social welfare as a function of the electoral rule, characterized by the parameter λ. The following proposition shows that, in contrast to the assertion that proportional rule fosters turnout because it makes all votes count, turnout is actually increasing in the weight the electoral system assigns to majority rule. Moreover, higher turnout leads to higher social welfare. It follows that, if the electorate is evenly split in expectation, both turnout and welfare are maximized by adopting majority rule.

Proposition 2. *At* $\alpha = \frac{1}{2}$ *turnout and welfare are both increasing in the weight* λ *assigned to majority rule.*

Let us first discuss the turnout result. Majority rule makes votes relevant only if they are pivotal, that is, if they help to break a tie or reach one. This may be a very low-probability event, but the impact of such pivotal vote is very high. On the other hand, proportional rule ensures that all votes in favor of an alternative help increase its winning probability. However, the magnitude of that impact is relatively small. Proposition 2 shows that in this trade off between frequency and magnitude of a vote's impact, the latter is a stronger force. Majority rule gives voters a higher ex ante incentive to cast a ballot and that results in higher turnout than under proportional rule.

As stated in the above proposition welfare also increases as the weight assigned to majority rule increases. This result is somewhat counterintuitive, as it disproves our initial concern about the "tyranny of the majority" problem. A social planner wishing to maximize welfare would deny any representation to the losing alternative. This is because ex ante social welfare is given by the sum of the expected utility of those citizens who abstain and those who vote. When voters are equally likely to support the two alternatives, the expected utility of a citizen who abstains under

proportional rule is the same as under majority rule, i.e. $\frac{1}{2}$. It follows that welfare depends solely on the expected benefit from going to the polls, which is maximized by majority rule.

We now turn our attention to the case in which the two parties are expected to be of similar, but not identical, size. Unfortunately, many of the properties of the symmetric equilibrium are lost when we move away from the case where the electorate is evenly split in expectation. As it is not possible to obtain explicit analytic solutions for the equilibrium thresholds for a generic value of α, we cannot perform straightforward comparative statics on turnout and welfare as a function of both α and λ. However, we can analyze what happens in a neighborhood of $\alpha = \frac{1}{2}$ by applying a continuity argument. Indeed, we can prove that in a neighborhood of $\alpha = \frac{1}{2}$ there exists a unique equilibrium such that the cutoffs are continuous in α (see Lemma 3 in the Appendix). This is useful in two ways. First, equilibrium uniqueness allows us to perform meaningful comparative statics. Second, since the equilibrium values are continuous functions of α, we can extend the turnout and welfare properties that hold at $\alpha = \frac{1}{2}$ to a neighborhood. Indeed, recall that both turnout and welfare are continuous in c_A^* and c_B^* and, therefore, they are also continuous in α.

Proposition 3. *In a neighborhood of $\alpha = \frac{1}{2}$, turnout and welfare are increasing in the weight λ assigned to majority rule.*

We started our analysis of tight elections armed with two opposing intuitions. On the one hand, the case where the expected size of the two groups is similar would seem the most likely to lead to a "tyranny of the majority" scenario, where almost half of the population sees their preferred alternative defeated. Hence, proportional rule would look like an appropriate candidate to mitigate that potential welfare loss. On the other hand, the argument that PR would encourage turnout by making every vote count seems rather weak when the election is expected to be close. The first intuition was revealed to be wrong. This is because, when elections are tight, proportional rule not only does not encourage turnout, but actually depresses it, even among minority supporters. Although it increases the frequency of a vote having positive impact compared with majority rule, the magnitude of that impact is far too small, resulting in a lower expected benefit from voting. In turn, social welfare is directly determined by voter turnout. As the expected benefit of abstainers is insensitive to the electoral rule, welfare depends solely on the benefit from voting. Thus, welfare is maximized under majority rule because such a system encourages voters of both groups to vote more frequently than under proportional rule.

4. Biased elections

In this section, we focus on the case where the expected size of the minority group is rather small. This is the scenario where PR, in principle, is more likely to remedy the problem of low incentives for minority supporters that emerges under FPTP. At the same time, it is also the case where the "tyranny of the majority" would seem

less worrisome, since the victory of the majoritarian alternative would benefit a large fraction of the population. This second factor would run in favor of adopting majority rule.

We begin by analyzing the extreme case where all citizens support the majoritarian alternative, i.e. $\alpha = 1$. Clearly, as everyone supports alternative A, the equilibrium will be fully symmetric. The next lemma proves its uniqueness.

Lemma 2. *When all citizens support the majoritarian alternative, i.e. $\alpha = 1$, there exists a unique symmetric BNE such that $c_A^* \epsilon (\underline{c}, c)$.*

Note that, differently from the case of $\alpha = \frac{1}{2}$, it cannot happen that all citizens turn out to vote in equilibrium. Indeed, as everyone supports the same alternative, if anyone else voted with probability 1 an individual would be better off by abstaining. At this equilibrium, turnout and social welfare are invariant across all voting rules in the class Ω_λ (see the proof of Lemma 2 for a formal argument). The proof of this result rests on showing that when $\alpha = 1$ the benefit from voting under any electoral rule q_λ is identical to the benefit from voting under majority rule. Since everybody supports the same alternative, even under proportional rule a voter can only make a difference when everyone else abstains, in which case a decisive vote has the same impact independently of the electoral rule. Thus, turnout does not vary with λ. Moreover, the expected benefit from abstaining is also independent of λ. These two factors imply that social welfare is also constant across all electoral rules. Hence, while it is actually true that majority rule does not lead to the "tyranny of the majority", it is also true that welfare is not diminished by adopting any other electoral rule.

Analogously to what we did for the case of tight elections, we can extend equilibrium uniqueness to a neighborhood of $\alpha = 1$ and show that the equilibrium values are continuous in α (see Lemma 4 in the Appendix). It follows that, in a neighborhood of $\alpha = 1$, turnout and welfare are continuous in α. Since turnout and welfare are identical under all voting rules when everyone supports A, this means that in a tight election they will be relatively insensitive to the electoral rule. Unfortunately though, this also implies that we cannot infer any comparative statics properties for a neighborhood of $\alpha = 1$ by applying a continuity argument. However, we can compare the individual turnout probability within each group of supporters, and analyze how they vary as a function of the weight λ given to majority rule.

Proposition 4. *In a neighborhood of $\alpha = 1$, the unique type-symmetric BNE has the following properties.*

(i)   *Underdog effect: for any voting rule q_λ, individual turnout probability is greater among minority supporters than among majority supporters.*

(ii) *Make minorities count: the individual turnout probability of minority supporters is decreasing in the weight λ assigned to majority rule.*

This proposition shows that, although in a biased election turnout is quite insensitive to the electoral rule, members of the minority group vote relatively more often than majority supporters. This is a consequence of the greater collective

action problem faced by the latter. For the case of majority rule, this phenomenon is typically referred to as the *underdog effect* (Levine and Palfrey, 2007). Several studies provide empirical evidence in its support (see, for example, Blais, 2000; Shachar and Nalebuff, 1999). The underdog effect has been formalized in a general setting by Taylor and Yildirim (2010) and Myatt (2012) shows that under majority rule and aggregate uncertainty over the size of the majority group the effect is so strong that the election ties in expectation. We extend this result to the use of any electoral rule q_λ.

The second part of the proposition shows that minority supporters vote more often the more proportional the electoral rule is. Hence, in biased elections at least, the argument put forward by the "Make My Vote Count" initiative is indeed correct. To see this, let us focus on a member of the minority group. As we know, under proportional rule her vote always has a positive impact, albeit of modest magnitude. Under majority rule, the impact is only positive if she is pivotal. However, in a biased election this event occurs with an extremely low frequency, much lower than when the electorate is evenly split. As a result, contrary to the case of tight elections, frequency is the determinant force of voting decisions. By making all votes count, more proportional rules give minority supporters greater incentives to vote.

Although explicit calculations are not possible, the combination of these two results suggest that in biased elections, just like in tight ones, social welfare must be higher the more weight is assigned to majority rule. This is because the higher λ, the lower the individual turnout probability within the minority group. This in turn weakens the underdog effect and makes the minoritarian alternative less likely to win.

5. Comparing tight and biased elections

So far we have focused on two polar cases, elections where the electorate is expected to be almost evenly split and elections where the expected size of the minority group is close to zero. We have not yet made any comparisons between these two cases. Specifically we wish to answer the two following questions. First, how does turnout compare between tight and biased elections? This is a very relevant issue, which is often studied in empirical analyses of voter turnout (Blais, 2006). Second, under proportional representation, do minority supporters vote more frequently in biased or in tight elections?

The following proposition characterizes our results on these issues.

Proposition 5 (Generalized and reversed competition effects) *Turnout in a tight election is always higher than in a biased election. Under proportional rule, the individual turnout probability within the minority group is higher in a biased election than in a tight election.*

The first part of this proposition shows that election closeness, measured by citizen's ex ante probability of supporting the minoritarian alternative $1-\alpha$, has a positive impact on turnout. In the case of majority rule, this phenomenon is known as

the *competition effect* (Levine and Palfrey, 2007) and it is consistent with the existing empirical evidence on real elections. For instance, Shachar and Nalebuff (1999) find a positive effect of predicted closeness in turnout in US presidential elections, while Noury (2004) finds a similar effect in roll call votes in the European Parliament. Levine and Palfrey (2007) observe this phenomenon in the laboratory as well. We generalize the competition effect in two directions. First, we show that it holds for any given electoral rule in the family Ω_λ. Second, we prove that it holds even when comparing different voting rules across the two scenarios: the turnout under any rule in a tight election is higher than the highest possible turnout in a biased election.

This result also implies that turnout is more sensitive to election closeness the more weight is assigned to majority rule. This is because, as we know from Propositions 2 and 5, turnout is fostered by majority rule when elections are tight, whereas it is relatively insensitive to the voting rule when they are biased. The intuition behind this result is that proportional rule provides voters with flatter incentives than majority rule. The impact of a vote under proportional rule is always positive but relatively small. Hence, election closeness has a more limited effect on turnout the more proportional the electoral rule is.

The second part of the proposition shows that under proportional rule turnout within the minority group is higher when the election is biased than when the election is tight. This *reversed competition effect* is due to the differential marginal effect of a minority vote in these two scenarios. Under proportional rule, the share of the prize q_P is a concave function of the number of votes. Hence, the magnitude of impact when the expected size of the minority group tends to zero is larger than when the minority group comprises about half of the population. This higher magnitude of impact fosters turnout within the minority group more in biased elections than in tight ones, generating this reversed competition effect which Herrera et al. (2014) find in their laboratory experiment.

6. Conclusion

The debate on the best electoral rule is a long-lasting one. From the works of John Stuart Mill (1861) and Thomas Hare (1865) to initiatives like the "Make My Vote Count" campaign, many have defended PR against majority rule as a rule that enhances welfare, fosters turnout and protects minorities. In this paper, we have studied how different electoral rules affect turnout and social welfare. Our results suggest that most of the above arguments against majority rule do not hold true, at least in the context of a model of costly instrumental voting and under reasonable informational assumptions. We show that when the electorate is expected to be split, majority rule induces higher turnout and social welfare than proportional rule. And when the size of the minority group tends to zero, both turnout and social welfare are independent of the electoral rule employed.

These results run against the intuition behind the "tyranny of the majority" argument and seem at odds with the available empirical evidence. Studies on electoral turnout show that PR systems yield higher levels of participation than

majoritarian systems, at least in well established democracies (for a survey see Blais, 2006). However, in a laboratory experiment, Schram and Sonnemanns (1996) obtain the opposite. Our results suggest that this conflicting evidence may be due to the different effect that electoral rules have on overall turnout depending on the expected size of the minority group. Hence, empirical studies on the relation between electoral systems and turnout in real elections should control for expected closeness. Whereas the positive effect of expected closeness on turnout is well-described in the empirical literature (Blais, 2006), to the best of the authors' knowledge, there are no empirical studies measuring this effect in PR systems.

We also showed that overall turnout is higher in a tight election than in a biased one, under any electoral rule. This result holds true as well when different voting rules are applied in these two scenarios (*generalized competition effect*). We also paid extensive attention to voting decisions within the minority group. We showed that when elections are biased, minority supporters vote more often than majority supporters (*underdog effect*), and that this effect is strongest under proportional rule. Finally, we compared minority turnout between the two polar cases of tight elections and biased elections. We showed that the positive effect of proportional rule on the turnout of minority supporters is higher in the latter case, so the minority group experiences a *reversed competition effect*.

We should stress here that our results rest on a key feature of our approach, which is that voting is only instrumental. Hence, social welfare is measured by the expected benefit of voting, which in turn is a combination of the frequency and magnitude of a vote's impact. This is perhaps against the classical view on the "tyranny of the majority" issue, which implicitly places a high weight on the welfare of the defeated group. Our measure of social welfare, as it is customary in the literature, is based on utilitarian principles. Hence, a stronger underdog effect, due either to a biased election or a more proportional rule, tends to reduce welfare by decreasing the winning probability of the majoritarian alternative. However, it is worth noting that Kartal (2014) shows that PR can generate higher social welfare than FPTP when the minority group enjoys a cost advantage and voting is still purely instrumental.

One last remark is in order. We did not compare social welfare between tight and biased elections. This is because the final effect depends on the size of the electorate and the distribution of costs. Although citizens have a higher incentive to vote when the election is tight, a larger share of the electorate will see their preferred alternative defeated. When the election is biased, the collective action problem is strong within the majority group but the overwhelmingly majoritarian alternative is selected almost surely. These two opposing factors make comparison not clear-cut.

Funding

Notes

1. "In the large constituencies, nearly half of the electors are, for all useful purposes, in the same position as if they were disenfranchised" (Hare, 1865).
2. Toccqueville (1835) and Mill (1859) also referred to this problem, but rather than on this potential caveat of majority rule, they focused on the social pressure stemming from majoritarian opinions, which they called the "*moral* tyranny of the majority" or the "tyranny of the public sentiment".
3. Several systems are indeed a combination of proportional and majority rule: some parliament seats are awarded proportionally, while others are awarded in a majoritarian way, e.g. through a majority bonus.
4. There exist several examples of "mixed" electoral systems. For instance, in the general elections that were held in Italy between 1993 and 2005, 75% of the parliament seats were assigned according to FPTP and the remaining 25% according to PR. Similarly, since 2003, 73 of the 123 seats in the Scottish Parliament are assigned by FPTP and the remaining by PR.
5. There exist other forms of modeling electoral rules in a compact way. Herrera et al. (2014) use a function often employed in the contests literature and Kartal (2014) uses a smooth function as in Faravelli et al. (2013).

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Appendix

Proof of Proposition 1. For this to be an equilibrium, it must be the case that the supporter of A with cost $\hat{c}_A$ and the supporter of B with cost $\hat{c}_B$ must be indifferent between voting and abstaining, if each other player adopts this strategy. That is, $B_A(\hat{c}_A) = \hat{c}_A$ and $B_B(\hat{c}_B) = \hat{c}_B$.

To prove the existence of such an equilibrium, define the function $\Phi : [\underline{c}, \bar{c}]^2 \to [\underline{c}, c]^2$ as

$$\Phi(\hat{c}_A, \hat{c}_B) = (\max\{\min\{B_A(\hat{c}_A, \hat{c}_B), \bar{c}\}, \underline{c}\}, \max\{\min\{B_B(\hat{c}_A, \hat{c}_B), \bar{c}\}, \underline{c}\}).$$

As $B_z(\hat{c}_A, \hat{c}_B)$ is clearly continuous in $\hat{c}_A$ and $\hat{c}_B$ then the operator $\Phi(\hat{c}_A, \hat{c}_A)$ is also continuous. Therefore, by Brower's fixed point theorem there must exist a pair (c_A^*, c_B^*) such that $\Phi(c_A^*, c_B^*) = (c_A^*, c_B^*)$. If $c_z^* \in (\underline{c}, c)$ then all supporters of the generic alternative z with cost lower than c_z^* will vote, while those with a higher cost will

abstain. Similarly, if $c_z^* = \underline{c}$ all supporters of the alternative will abstain, while if $c_z^* = \bar{c}$ they will all vote.

Proof of Lemma 1. Conditional on all citizens abiding to strategy s, and given that each one of them is equally likely to support either A or B, it is possible to express the expected benefit of voting for a generic individual i as a function of the individual ex ante probability of voting, i.e., before learning her type, which is identical for all citizens. This probability is given by

$$\pi = \int_{\underline{c}}^{\hat{c}} s(c)f(c)\,dc.$$

Hence, the expected benefit from voting for a supporter of A, given by expression (3), can be written as a function of the ex ante voting probability π and boils down to

$$B(\pi) = \sum_{v=0}^{n} \binom{n}{v}\pi^v(1-\pi)^{n-v}\sum_{m=0}^{v}[q_\lambda(m+1,v-m)-q_\lambda(m,v-m)]\binom{v}{m}\frac{1^v}{2}. \quad (5)$$

Observe that, since citizens support either party with probability $\frac{1}{2}$, the expected benefit from voting is the same regardless of the alternative favored by an individual. We can thus drop the subindex z in expression (5) and denote by m the number of citizens who vote for the same alternative supported by citizen i. The term $\binom{n}{v}\pi^v(1-\pi)^{n-v}$ represents the probability that v of the n citizens other than i turn out to vote. The expression $[q_\lambda(m+1,v-m)-q_\lambda(m,v-m)]$ corresponds to the benefit from voting when m of the v voters are already voting for the same alternative supported by i, and it is multiplied by the probability of that event happening, which is equal to $\binom{v}{m}\frac{1^v}{2}$. Note that $B(0)=\frac{1}{2}$.

We can rearrange (5) as

$$
\begin{aligned}
B(\pi) = {}& (1-\lambda)\sum_{v=0}^{n}\binom{n}{v}\pi^v(1-\pi)^{n-v} \\
& \times \sum_{m=0}^{v}[q_P(m+1,v-m)-q_P(m,v-m)]\binom{v}{m}\frac{1^v}{2} \\
& + \lambda\sum_{v=0}^{n}\binom{n}{v}\pi^v(1-\pi)^{n-v} \\
& \times \sum_{m=0}^{v}[q_M(m+1,v-m)-q_M(m,v-m)]\binom{v}{m}\frac{1^v}{2}. \\
= {}& (1-\lambda)P(\pi)+\lambda M(\pi)
\end{aligned}
\quad (6)
$$

Hence, $B(\pi)$ is a convex combination of $M(\pi)$ and $P(\pi)$, where the former corresponds to the expected benefit from voting under majority rule and the latter is the expected benefit from voting under proportional rule.

Note that the term $M(\pi)$ in expression (6) corresponds to the expected benefit from voting under majority rule as analyzed by Börgers (2004). In Remark 1, he shows that $M(\pi)$ is differentiable and decreasing in π. It now remains to show that the same applies to $P(\pi)$, that is, the expected benefit from voting under proportional rule.

First, it is possible to express $P(\pi)$ as

$$P(\pi) = (1-\pi)^n \frac{1}{2} + \sum_{v=1}^{n} \binom{n}{v} \pi^v (1-\pi)^{n-v} \sum_{m=0}^{v} \binom{v}{m} \left[\frac{m+1}{v+1} - \frac{m}{v} \right] \frac{1^v}{2}$$

$$= (1-\pi)^n \frac{1}{2} + \sum_{v=1}^{n} \binom{n}{v} \pi^v (1-\pi)^{n-v} \sum_{m=0}^{v} \binom{v}{m} \left[\frac{v-m}{(v+1)v} \right] \frac{1^v}{2}.$$

Further, note that the last term $\sum_{m=0}^{v} \binom{v}{m} \left[\frac{v-m}{(v+1)v} \right] \frac{1^v}{2}$ can be expressed as

$$\sum_{m=0}^{v} \binom{v}{m} \left[\frac{v-m}{(v+1)v} \right] \frac{1^v}{2} = \sum_{m=0}^{v} \frac{v!}{m!(v-m)!} \left[\frac{v-m}{(v+1)v} \right] \frac{1^v}{2}$$

$$= \frac{1}{2(v+1)} \sum_{m=0}^{v} \binom{v-1}{m} \frac{1^{v-1}}{2}$$

$$= \frac{1}{2(v+1)} \sum_{m=0}^{v-1} \binom{v-1}{m} \frac{1^{v-1}}{2}$$

$$= \frac{1}{2(v+1)},$$

where the last equality holds by applying the binomial theorem. Hence, we can rewrite $P(\pi)$ as

$$P(\pi) = \frac{1}{2} \sum_{v=0}^{n} \binom{n}{v} \pi^v (1-\pi)^{n-v} \frac{1}{v+1} = \frac{1-(1-\pi)^{n+1}}{2\pi(n+1)}, \tag{7}$$

where the last equality follows from applying regular binomial rules.

Differentiating the above expression with respect to π we obtain the following:

$$\frac{\partial P(\pi)}{\partial \pi} = \frac{1}{2(n+1)} \left[-(n+1) \frac{(1-\pi)^n}{\pi} - \frac{1-(1-\pi)^{n+1}}{\pi^2} \right] < 0, \tag{8}$$

which implies that $P(\pi)$ is decreasing in π. As a consequence, we can conclude that $B(\pi)$ is also decreasing in π.

Given the cut-off strategy described previously, then $\pi = F(\hat{c})$. In addition, observe that $B(F(\underline{c})) = B(0) = \frac{1}{2} > \underline{c}$. Since $F(\hat{c})$ is increasing in $\hat{c}$, then $B(F(\hat{c}))$ is decreasing in $\hat{c}$ and there exists at most one cost level c^* such that $B(F(c^*)) = c^*$. This proves the existence of a unique equilibrium c^* such that $c^* \in (\underline{c}, \bar{c})$ if $B(F(\bar{c})) < \bar{c}$, while $c^* = \bar{c}$ otherwise.

Proof of Proposition 2. We begin by proving that turnout is increasing in λ. Recall from (6) that $B(\pi) = (1-\lambda)P(\pi) + \lambda M(\pi)$ and that

$$M(\pi) = \sum_{v=0}^{n} \binom{n}{v} \pi^v (1 - \pi)^{n-v}$$

$$\sum_{m=0}^{v} [q_M(m + 1, v - m) - q_M(m, v - m)] \binom{v}{m} \left(\frac{1}{2}\right)^v.$$

Note that, if v is odd, $q_M(m + 1, v-m) - q_M(m, v - m)$ is equal to $\frac{1}{2}$ when $m = \frac{v-1}{2}$ and zero otherwise. Similarly, if v is even, $q_M(m + 1, v - m) - q_M(m, v-m)$ is equal to $\frac{1}{2}$ when $m = \frac{v}{2}$ and equals zero otherwise. Finally, if $v = 0$ then $q_M(m + 1, v - m) - q_M(m, v - m) = \frac{1}{2}$. Let us define the function $g(v)$ as

$$g(v) = \begin{cases} 1 & \text{if } v = 0 \\ \binom{v}{\frac{v-1}{2}} & \text{if } v \text{ is odd,} \\ \binom{v}{\frac{v}{2}} & \text{if } v \text{ is even.} \end{cases}$$

We can now rewrite

$$M(\pi) = \frac{1}{2} \sum_{v=0}^{n} \binom{n}{v} \pi^v (1 - \pi)^{n-v} \left(\frac{1}{2}\right)^v g(v).$$

We want to compare $M(\pi)$ with $P(\pi)$, which we know from expression (7), as worked out in the proof of Lemma 1, that can be written as

$$P(\pi) = \frac{1}{2} \sum_{v=0}^{n} \binom{n}{v} \pi^v (1 - \pi)^{n-v} \frac{1}{v + 1}.$$

Hence,

$$M(\pi) - P(\pi) = \frac{1}{2} \sum_{v=0}^{n} \binom{n}{v} \pi^v (1 - \pi)^{n-v} \left[\left(\frac{1}{2}\right)^v g(v) - \frac{1}{v + 1} \right].$$

First, note that if $v = 0$, then $M(\pi) = P(\pi) = \frac{1}{2}$. Consider now the case of v odd. Note that in that case

$$\left(\frac{1}{2}\right)^{v} g(v) = \left(\frac{1}{2}\right)^{v} \binom{v}{\frac{v-1}{2}}$$

$$= \left(\frac{1}{2}\right)^{v} \frac{v!}{\left(\frac{v-1}{2}\right)! \left(\frac{v-1}{2}\right)! \frac{v+1}{2}}$$

$$= \left(\frac{1}{2}\right)^{v} \frac{v!}{\left(\frac{1}{2}\right)^{\frac{v+1}{2}} \left(\frac{1}{2}\right)^{\frac{v-1}{2}} (v+1)[(v-1)\times(v-3)\times\cdots\times 2]^{2}}$$

$$= \frac{1}{(v+1)} \frac{v!}{(v-1)^{2}\times(v-3)^{2}\times\cdots\times 2^{2}}$$

$$= \frac{1}{(v+1)} \frac{v\times(v-2)\times\cdots\times 3}{(v-1)\times(v-3)\times\cdots\times 2} > \frac{1}{(v+1)}.$$

Finally, consider v even. In this case

$$\left(\frac{1}{2}\right)^{v} g(v) = \left(\frac{1}{2}\right)^{v} \binom{v}{\frac{v}{2}}$$

$$= \left(\frac{1}{2}\right)^{v} \frac{v!}{\frac{v}{2}! \frac{v}{2}!}$$

$$= \left(\frac{1}{2}\right)^{v} \frac{v!}{\left(\frac{1}{2}\right)^{v}[v\times(v-2)\times\cdots\times 2]^{2}}$$

$$= \frac{1}{v} \frac{(v-1)\times(v-3)\times\cdots\times 3}{(v-2)\times(v-4)\times\cdots\times 2} > \frac{1}{v} > \frac{1}{v+1}.$$

Thus, we can conclude that $M(\pi) > P(\pi)$ and therefore $\frac{\partial B(\pi)}{\partial \lambda} > 0$. It follows that c^{*} and turnout are also increasing in λ.

We continue the proof by demonstrating that welfare is also increasing in λ. We begin by showing that under both majority and proportional rule the expected utility of a citizen who abstains is $\frac{1}{2}$. Börgers (2004) showed this for majority rule. This is rather intuitive since ex ante individuals are equally likely to support each alternative and the benefit in case of supporting the winning alternative is just 1. The case of proportional rule is not that straightforward. Although supporters of both alternatives will cast a vote with the same probability π, the benefit from supporting an alternative depends on its vote share. Hence it is necessary to consider all possible realizations of the vote shares.

When $\alpha = \frac{1}{2}$, the expected payoff for any citizen who abstains and supports alternative z, given by expression (2), boils down to

$$U(\pi) = \sum_{v=0}^{n} \binom{n}{v} \pi^{v}(1-\pi)^{n-v} \sum_{m=0}^{v} q_{P}(m, v-m) \binom{v}{m} \left(\frac{1}{2}\right)^{v}$$

$$= \sum_{v=0}^{n} \sum_{m=0}^{v} \binom{n}{v} \pi^{v}(1-\pi)^{n-v} \binom{v}{m} \left(\frac{1}{2}\right)^{v} \frac{m}{v},$$

which is the probability of the alternative supported by citizen i winning when v out of n citizens vote and m of them vote for that alternative, multiplied by the probability of that event, added up across all possible realizations of v and m. Note that when v voters vote, the probability that m do it in favor of i's favorite alternative is identical to the probability of $v - m$ doing the same. Hence,

$$U(\pi) = \sum_{v=0}^{n} \sum_{m=0}^{v} \binom{n}{v} \pi^v (1-\pi)^{n-v} \binom{v}{m} \left(\frac{1}{2}\right)^v \frac{m}{v} = \sum_{v=0}^{n} \binom{n}{v} \pi^v (1-\pi)^{n-v} h(v),$$

$$(9)$$

where

$$h(v) = \begin{cases} \sum_{m=0}^{\frac{v-1}{2}} \binom{v}{m} \left(\frac{1}{2}\right)^v & \text{if } v \text{ odd} \\ \sum_{m=0}^{\frac{v}{2}-1} \binom{v}{m} \left(\frac{1}{2}\right)^v + \binom{v}{\frac{v}{2}} \left(\frac{1}{2}\right)^{v+1} & \text{if } v \text{ even} \end{cases},$$

which in turn is equal to $\frac{1}{2}$. Thus, expression (9) reduces to

$$\frac{1}{2} \sum_{v=0}^{n} \binom{n}{v} \pi^v (1-\pi)^{n-v} = \frac{1}{2},$$

meaning that the expected utility from abstaining under proportional rule is $\frac{1}{2}$ regardless of the alternative the citizen supports. Given that q_λ is linear, this property extends to any intermediate value of λ.

Hence, expression (4) denoting an individual's ex ante expected welfare boils down to

$$W(\pi) = \int_{\underline{c}}^{c^*} \left(\frac{1}{2} + B(\pi) - c\right) f(c)\, dc + \int_{c^*}^{\bar{c}} \frac{1}{2} f(c)\, dc$$

$$= \int_{\underline{c}}^{c^*} \left(\frac{1}{2} + \lambda M(\pi) + (1-\lambda)P(\pi) - c\right) f(c)\, dc + \int_{c^*}^{\bar{c}} \frac{1}{2} f(c)\, dc$$

$$= \frac{1}{2} + \int_{\underline{c}}^{c^*} \left(\lambda M(\pi) + (1-\lambda)P(\pi) - c\right) f(c)\, dc.$$

Recall that $\pi = F(c^*)$. The function $W(\pi)$ is clearly differentiable with respect to c^*, so its derivative with respect to λ is

$$W'(\pi) = \int_{\underline{c}}^{c^*} B'(\pi) f(c^*) \frac{\partial c^*}{\partial \lambda} f(c)\, dc + (B(\pi) - c^*) f(c^*) \frac{\partial c^*}{\partial \lambda}$$

$$= B'(\pi) f(c^*) F(c^*) \frac{\partial c^*}{\partial \lambda}.$$

As we have proven that $B(\pi)$ and c^* are increasing in λ, social welfare $W(\pi)$ is also increasing in λ. ∎

Lemma 3. *In a neighborhood of $\alpha = \frac{1}{2}$, there exists a unique type-symmetric BNE. Moreover, the equilibrium cutoffs c_A^* and c_B^* are continuous functions of α.*

Proof. Consider a sequence α_m such that $\lim_{m \to \infty} \alpha_m = \frac{1}{2}$ and call c_A^m and c_B^m the corresponding equilibrium thresholds. By compactness, we know that there exist subsequences converging to some cutoffs $\tilde{c}_A$ and $\tilde{c}_B$. Moreover, since the benefit functions are continuous, the pair of cutoffs $(\tilde{c}_A, \tilde{c}_B)$ is an equilibrium for the case of $\alpha = \frac{1}{2}$. As the equilibrium is unique when $\alpha = \frac{1}{2}$, for any neighborhood of the $\alpha = \frac{1}{2}$ equilibrium there exists a critical value $\tilde{\alpha} > \frac{1}{2}$ such that all equilibria corresponding to $\alpha < \tilde{\alpha}$ are in that neighborhood. Having established this, we can apply the implicit function theorem to prove that the equilibrium is unique in a neighborhood of $\alpha = \frac{1}{2}$, and that the corresponding cutoffs are continuous functions of α.

Let us calculate the Jacobian in $\alpha = \frac{1}{2}$:

$$
J = \begin{pmatrix}
\frac{\partial B_A(c_A^*, c_B^*)}{\partial c_A^*}\Big|_{\alpha = \frac{1}{2}} - 1 & \frac{\partial B_A(c_A^*, c_B^*)}{\partial c_B^*}\Big|_{\alpha = \frac{1}{2}} \\
\frac{\partial B_B(c_A^*, c_B^*)}{\partial c_A^*}\Big|_{\alpha = \frac{1}{2}} & \frac{\partial B_B(c_A^*, c_B^*)}{\partial c_B^*}\Big|_{\alpha = \frac{1}{2}} - 1
\end{pmatrix}.
$$

Note that, by symmetry, $\frac{\partial B_A(c_A^*, c_B^*)}{\partial c_A^*}\Big|_{\alpha = \frac{1}{2}} = \frac{\partial B_B(c_A^*, c_B^*)}{\partial c_B^*}\Big|_{\alpha = \frac{1}{2}}$ and $\frac{\partial B_B(c_A^*, c_B^*)}{\partial c_A^*}\Big|_{\alpha = \frac{1}{2}} = \frac{\partial B_A(c_A^*, c_B^*)}{\partial c_B^*}\Big|_{\alpha = \frac{1}{2}}$. Hence, we can express the Jacobian determinant as

$$
\det J = \left(\frac{\partial B_A(c_A^*, c_B^*)}{\partial c_A^*}\Big|_{\alpha = \frac{1}{2}} - 1\right)^2 - \left(\frac{\partial B_A(c_A^*, c_B^*)}{\partial c_B^*}\Big|_{\alpha = \frac{1}{2}}\right)^2.
$$

If the determinant is non-zero, then J is invertible and we can apply the implicit function theorem. Note that $\det J \neq 0$ if and only if

$$
\frac{\partial B_A(c_A^*, c_B^*)}{\partial c_A^*}\Big|_{\alpha = \frac{1}{2}} - \frac{\partial B_A(c_A^*, c_B^*)}{\partial c_B^*}\Big|_{\alpha = \frac{1}{2}} \neq 1.
$$

Denote by v_A the number of votes in favor of alternative A. When $\alpha = \frac{1}{2}$ the expected benefit (3) reduces to

$$
B_A(c_A^*, c_B^*) = \frac{1}{2^n} \sum_{v=0}^{n} \sum_{v_A=0}^{v} \binom{n}{v}\binom{v}{v_A} F(c_A^*)^{v_A} F(c_B^*)^{v-v_A} [2 - F(c_A^*) - F(c_B^*)]^{n-v}
$$

$$
\times [q_\lambda(v_A + 1, v - v_A) - q_\lambda(v_A, v - v_A)].
$$

(10)

Differentiating (10) with respect to c_A^* we obtain the following:

$$\frac{\partial B_A(c_A^*, c_B^*)}{\partial c_A^*}\bigg|_{\alpha = \frac{1}{2}} = \frac{1}{2^n} f(c_A^*) \sum_{v=0}^{n} \sum_{v_A=0}^{v} \binom{n}{v} \binom{v}{v_A} F(c_A^*)^{v_A-1} F(c_B^*)^{v-v_A}$$

$$\times [2 - F(c_A^*) - F(c_B^*)]^{n-v-1}$$
$$\times v_A [2 - F(c_A^*) - F(c_B^*)] - (n-v)F(c_A^*)]$$
$$\times [q_\lambda(v_A + 1, v - v_A) - q_\lambda(v_A, v - v_A)].$$

As we know that $F(c_A^*) = F(c_B^*)$ when $\alpha = \frac{1}{2}$, we can rewrite the above expression as

$$\frac{\partial B_A(c_A^*, c_B^*)}{\partial c_A^*}\bigg|_{\alpha = \frac{1}{2}}$$

$$= f(c^*) \sum_{v=0}^{n} \sum_{v_A=0}^{v} \binom{n}{v} \binom{v}{v_A} \frac{1}{2^v} F(c^*)^{v-1} [1 - F(c^*)]^{n-v-1} \quad (11)$$

$$\times [v_A[1 - F(c^*)] - (n-v)F(c^*)]$$
$$\times [q_\lambda(v_A + 1, v - v_A) - q_\lambda(v_A, v - v_A)].$$

Similarly, differentiating $B_A(c_B^*, c_B^*)$ with respect to c_B^* and calculating the derivative $\alpha = \frac{1}{2}$ we get the following

$$\frac{\partial B_A(c_A^*, c_B^*)}{\partial c_B^*}\bigg|_{\alpha = \frac{1}{2}}$$

$$= f(c^*) \sum_{v=0}^{n} \sum_{v_A=0}^{v} \binom{n}{v} \binom{v}{v_A} \frac{1}{2^v} F(c^*)^{v-1} [1 - F(c^*)]^{n-v-1} \quad (12)$$

$$\times [(v - v_A)[1 - F(c^*)] - (n-v)F(c^*)]$$
$$\times [q_\lambda(v_A + 1, v - v_A) - q_\lambda(v_A, v - v_A)].$$

Call Δ the difference between (11) and (12). Then

$$\Delta = f(c^*) \sum_{v=0}^{n} \sum_{v_A=0}^{v} \binom{n}{v} \binom{v}{v_A} \frac{1}{2^v} F(c^*)^{v-1} [1 - F(c^*)]^{n-v}$$

$$\times (2v_A - v) \quad (13)$$
$$\times [q_\lambda(v_A + 1, v - v_A) - q_\lambda(v_A, v - v_A)].$$

We want to study the sign of Δ. First consider the case of $\lambda = 1$, that is, when majority rule is applied. Recall that $q_M(v_A + 1, v - v_A) - q_M(v_A, v - v_A)$ is either equal to zero or to $\frac{1}{2}$. Moreover, note that this difference is equal to $\frac{1}{2}$ either when $2v_A = v$ or when $2v_A = v - 1$ (for strictly positive values of v). But, in these cases, the second term of expression (13) is either zero or strictly negative, meaning that $\Delta < 0$ when majority rule is applied.

Let us now explore the case of $\lambda = 0$, i.e., when proportional rule is employed. We can safely ignore the case $v = 0$, as $2v_A - v$ would also be equal to zero. First, consider v even and let $i < \frac{v}{2}$. Notice that $\binom{v}{i} = \binom{v}{v-1}$. Hence, for a given v, the cases $v_A = i$ and $v_A = v - i$ are multiplied by the same binomial coefficient and, consequently, in both cases the first line of expression (13) assumes the same value. Further notice that $q_P(v_A + 1, v - v_A) - q_P(v_A, v - v_A) = \frac{v - v_A}{(v+1)v}$. Therefore, the third term of (13) is greater when $v_A = i$ than when $v_A = v - i$. But, when $v_A = i$ the term $(2v_A - v)$ is negative. Vice versa, $(2v_A - v)$ is positive when $v_A = v - i$. This means that we are multiplying greater values times negative numbers and smaller values times positive numbers. Finally notice that $(2v_A - v)$ is equal to zero when $v_A = \frac{v}{2}$, resulting in Δ being strictly negative. When v is odd, the same reasoning carries through, with the only difference that, instead of $v_A = \frac{v}{2}$, we now have to consider the case $v_A = \frac{v}{2} + 1$. Notice that in this case $(2v_A - v)$ assumes a negative value.

If Δ is negative both when $\lambda = 1$ and when $\lambda = 0$, then it is clearly negative for any λ. Having proved that $\Delta < 0$ assures that the Jacobian determinant is different from zero. Hence, the implicit function theorem guarantees that in a neighborhood of $\alpha = \frac{1}{2}$ the equilibrium is unique, and that the equilibrium thresholds c_A^* and c_B^* are continuous functions of α. ∎

Proof of Lemma 2. When $\alpha = 1$ a strategy can be fully described by a single cut-off cost $\hat{c}_A$. The expected benefit from voting for an individual is given by

$$B_A(\hat{c}_A) = \sum_{v=0}^{n} \binom{n}{v} F(\hat{c}_A)^v [1 - F(\hat{c}_A)]^{n-v} [q_\lambda(v + 1, 0) - q_\lambda(v, 0)]$$

$$= \lambda \sum_{v=0}^{n} \binom{n}{v} F(\hat{c}_A)^v [1 - F(\hat{c}_A)]^{n-v} [q_M(v + 1, 0) - q_M(v, 0)]$$

$$+ (1 - \lambda) \sum_{v=0}^{n} \binom{n}{v} F(\hat{c}_A)^v [1 - F(\hat{c}_A)]^{n-v} [q_P(v + 1, 0) - q_P(v, 0)].$$

By definition, whenever $v > 0$ it is the case that $q_M(v + 1, 0) - q_M(v, 0) = q_P(v + 1, 0) - q_P(v, 0) = 0$. On the other hand, when $v = 0$ we have $q_M(v + 1, 0) - q_M(v, 0) = q_P(v + 1, 0) - q_P(v, 0) = \frac{1}{2}$. Thus, the benefit from voting boils down to

$$B_A(\hat{c}_A) = \frac{1}{2}[1 - F(\hat{c}_A)]^n. \tag{14}$$

Since $B_A(\hat{c}_A)$ is non-increasing in $\hat{c}_A$ there must exist a unique cost level c_A^* such that $B_A(c_A^*) = c_A^*$. Note that $\frac{1}{2}[1 - F(\underline{c})]^n = \frac{1}{2} > \underline{c}$ and $\frac{1}{2}[1 - F(\bar{c})]^n = 0 < \bar{c}$, implying that $\underline{c} < c_A^* < \bar{c}$.

We immediately realize from expression (14) that the expected benefit from voting does not depend on λ, implying that turnout is identical under all voting rules.

Note that this means that the expected benefit of a voter conditional on abstaining is also independent of λ. Indeed, for any voting rule q_λ, an abstainer receives a benefit equal to 1 when $v > 0$ and equal to 0 when $v = 0$; since c_A^* is constant, these two events happen with the same probability across all rules. As the expected benefit from voting and from abstaining is invariant across all electoral rules, social welfare must also be invariant.

Lemma 4. *In a neighborhood of $\alpha = 1$, there exists a unique type-symmetric BNE. Moreover, the equilibrium cutoffs c_A^* and c_B^* are continuous functions of α.*

Proof. Consider a strategy defined by (1) and thresholds $\hat{c}_A$ and $\hat{c}_B$. We know that for this to be an equilibrium it must be the case that $\hat{c}_A = B_A(\hat{c}_A, \hat{c}_B)$ and $\hat{c}_B = B_B(\hat{c}_A, \hat{c}_B)$. In Proposition 3, Krasa and Polborn (2009) demonstrate that: (1) the equilibrium under majority rule is unique in a neighborhood of $\alpha = 1$ and (2) the equilibrium values c_A^* and c_B^* are continuous in α. Since majority rule and our generic electoral rule q_λ coincide at $\alpha = 1$, both of these results also extend to our setup. ■

Proof of Proposition 4. Let us start by considering the benefit from voting for a supporter of A. By continuity, we know from (14) that when α that tends to 1 the benefit from voting for an A -member is given by

$$\lim_{\alpha \to 1} B_A(\hat{c}_A, \hat{c}_B) = \frac{1}{2}[1 - F(\hat{c}_A)]^n. \tag{15}$$

Hence, when $\alpha \to 1$ the equilibrium cut-off c_A^* is found by solving the following equation in one unknown

$$\frac{1}{2}[1 - F(\hat{c}_A)]^n = \hat{c}_A.$$

Consider now $B_B(\hat{c}_A, \hat{c}_B)$, and note that it can be expressed as

$$
\begin{aligned}
B_B(\hat{c}_A, \hat{c}_B) = \lambda \sum_{v=1}^{n} \sum_{v_A=0}^{v} & \binom{n}{v}\binom{v}{v_A}[\alpha F(\hat{c}_A)]^{v_A}[(1-\alpha)F(\hat{c}_B)]^{v-v_A} \\
& \times [1 - \alpha F(\hat{c}_A) - (1-\alpha)F(\hat{c}_B)]^{n-v} \\
& \times [q_M(v - v_A + 1, v_A) - q_M(v - v_A, v_A)] \\
+ (1-\lambda) \sum_{v=1}^{n} \sum_{v_A=0}^{v} & \binom{n}{v}\binom{v}{v_A}[\alpha F(\hat{c}_A)]^{v_A}[(1-\alpha)F(\hat{c}_B)]^{v-v_A} \\
& \times [1 - \alpha F(\hat{c}_A) - (1-\alpha)F(\hat{c}_B)]^{n-v}\frac{v_A}{(v+1)v} \\
+ \frac{1}{2}[1 &- \alpha F(\hat{c}_A) - (1-\alpha)F(\hat{c}_B)]^{n}.
\end{aligned}
$$

Taking the limit of the above expression for α that tends to 1 we obtain the following

$$\lim_{\alpha \to 1} B_B(\hat{c}_A, \hat{c}_B) = \lambda \frac{1}{2} n F(\hat{c}_A)[1 - F(\hat{c}_A)]^{n-1}$$

$$+ (1 - \lambda) \sum_{v=1}^{n} \binom{n}{v} F(\hat{c}_A)^v [1 - F(\hat{c}_A)]^{n-v} \frac{1}{v+1} \qquad (16)$$

$$+ \frac{1}{2}[1 - F(\hat{c}_A)]^n.$$

Just like in the case of A-supporters, supporters of B receive a benefit of $\frac{1}{2}$ from voting when everybody else abstains. This is true under both proportional and majority rule and is captured by the last term of the above expression. That aside, under majority rule, as $\alpha \to 1$, a B-supporter can only be pivotal when only one other individual votes, as represented by the first term of expression (16). Finally, the second term captures the proportional component of the benefit. However, note that when only another person votes the benefit under proportional rule is identical to the benefit under majority rule. Hence, we can rewrite (16) as

$$\lim_{\alpha \to 1} B_B(\hat{c}_A, \hat{c}_B) = \frac{1}{2} n F(\hat{c}_A)[1 - F(\hat{c}_A)]^{n-1}$$

$$+ (1 - \lambda) \sum_{v=2}^{n} \binom{n}{v} F(\hat{c}_A)^v [1 - F(\hat{c}_A)]^{n-v} \frac{1}{v+1} \qquad (17)$$

$$+ \frac{1}{2}[1 - F(\hat{c}_A)]^n.$$

From this we can draw two conclusions. First, note that, just like (15), the above expression is a function of $\hat{c}_A$ alone. Moreover, note that (17) is strictly greater than (15) for any λ. Since we know that $c_A^* < \bar{c}$, it follows that when $\alpha \to 1$ it must be that $c_B^* > c_A^*$ for any λ. The continuity of the equilibrium cutoffs guarantees the existence of a neighborhood of $\alpha = 1$ such that $F(c_B^*) > F(c_A^*)$. Second, because expression (17) is decreasing in λ, then c_B^* is also decreasing in λ when $\alpha \to 1$. Since $B_B(\hat{c}_A, \hat{c}_B)$ is continuous in α there exists a neighborhood of $\alpha = 1$ such that $\frac{\partial B_B(\hat{c}_A, \hat{c}_B)}{\partial \lambda} < 0$ and, therefore, such that $F(c_B^*)$ is decreasing in λ.

Proof of Proposition 5. Let us prove the first part of the proposition. We start by comparing turnout at $\alpha = \frac{1}{2}$ and $\alpha = 1$. As we know, when $\alpha = \frac{1}{2}$ a strategy can be described by a single cut-off cost. Let us focus on A-supporters and thus call the cut-off $\hat{c}_A$. Recall that at $\alpha = \frac{1}{2}$ turnout is lowest under proportional rule q_P, in which case the equilibrium threshold $c_A^{\alpha = \frac{1}{2}, q_P}$ is given by the solution to the following equation

$$\frac{1}{2}[1 - F(\hat{c}_A)]^n + \frac{1}{2} \sum_{v=1}^{n} \binom{n}{v} F(\hat{c}_A)^v [1 - F(\hat{c}_A)]^{n-v} \frac{1}{v+1} = \hat{c}_A. \qquad (18)$$

When $\alpha = 1$, everyone supports A and the equilibrium cut-off $c_A^{\alpha = 1}$ is found by solving the following

$$\frac{1}{2}[1 - F(\hat{c}_A)]^n = \hat{c}_A, \qquad (19)$$

independently of the voting rule q_λ.

First, note that the left-hand side of (18) is strictly greater than the left-hand side of (19) for any $\hat{c}_A > \underline{c}$, while they coincide when $\hat{c}_A = \underline{c}$. Further, we know from expression (8), as worked out in the proof of Lemma 1, that the left-hand side of (18) is decreasing in $\hat{c}_A$, as it is clearly the case for the left-hand side of (19). Since we know from Proposition 2 that $\underline{c} < c_A^{\alpha=1} < \bar{c}$, it follows that $c_A^{\alpha=\frac{1}{2}, q_P} > c_A^{\alpha=1}$ and thus $F\left(c_A^{\alpha=\frac{1}{2}, q_P}\right) > F\left(c_A^{\alpha=1}\right)$. At $\alpha = \frac{1}{2}$, in equilibrium supporters of alternative B follow the same strategy adopted by A-supporters. On the other hand, at $\alpha = 1$ there are no supporters of B. Hence, turnout at $\alpha = \frac{1}{2}$ under q_P is higher than turnout at $\alpha = 1$ under any voting rule q_λ. Since the former is a lower bound, we can conclude that turnout at $\alpha = \frac{1}{2}$ is always higher than turnout at $\alpha = 1$, independently of the voting rule. Finally, we know that turnout is a continuous function of α, both in a neighborhood of $\alpha = \frac{1}{2}$ and $\alpha = 1$. Hence, it must be the case that the lowest possible turnout in a tight election, i.e. under q_P, is higher than the maximum turnout in a biased election.

Let us now prove the second part of the proposition. We focus on the voting rule q_P, hence we drop the superscript to simplify notation. We want to compare the individual turnout probability of supporters of alternative B in a neighborhood of $\alpha = \frac{1}{2}$ and $\alpha = 1$. Call $c_B^{\alpha=\frac{1}{2}}$ and $c_B^{\alpha \to 1}$ the equilibrium cut-off for supporters of B when $\alpha = \frac{1}{2}$ and when $\alpha \to 1$, respectively.

On the one hand, we know by symmetry that $c_B^{\alpha=\frac{1}{2}} = c_A^{\alpha=\frac{1}{2}}$ and therefore we can write

$$c_B^{\alpha=\frac{1}{2}} = \frac{1}{2}\left[1 - F\left(c_A^{\alpha=\frac{1}{2}}\right)\right]^n + \frac{1}{2}\sum_{v=1}^{n}\binom{n}{v}F\left(c_A^{\alpha=\frac{1}{2}}\right)^v\left[1 - F\left(c_A^{\alpha=\frac{1}{2}}\right)\right]^{n-v}\frac{1}{v+1}.$$

On the other hand, we know from (16) that when $\alpha \to 1$ then B's equilibrium cut–off is equal to

$$c_B^{\alpha \to 1} = \frac{1}{2}\left[1 - F\left(c_A^{\alpha=1}\right)\right]^n + \sum_{v=1}^{n}\binom{n}{v}F\left(c_A^{\alpha=1}\right)^v\left[1 - F\left(c_A^{\alpha=1}\right)\right]^{n-v}\frac{1}{v+1}.$$

Recall that $c_A^{\alpha=\frac{1}{2}} > c_A^{\alpha=1}$ and that the left-hand side of (18) is decreasing in $\hat{c}_A$. Therefore, it follows that

$$c_B^{\alpha=\frac{1}{2}} = \frac{1}{2}\left[1 - F\left(c_A^{\alpha=\frac{1}{2}}\right)\right]^n + \frac{1}{2}\sum_{v=1}^{n}\binom{n}{v}F\left(c_A^{\alpha=\frac{1}{2}}\right)^v\left[1 - F\left(c_A^{\alpha=\frac{1}{2}}\right)\right]^{n-v}\frac{1}{v+1}$$

$$< \frac{1}{2}\left[1 - F\left(c_A^{\alpha=\frac{1}{2}}\right)\right]^n + \sum_{v=1}^{n}\binom{n}{v}F\left(c_A^{\alpha=\frac{1}{2}}\right)^v\left[1 - F\left(c_A^{\alpha=\frac{1}{2}}\right)\right]^{n-v}\frac{1}{v+1}$$

$$< \frac{1}{2}\left[1 - F\left(c_A^{\alpha=1}\right)\right]^n + \sum_{v=1}^{n}\binom{n}{v}F\left(c_A^{\alpha=1}\right)^v\left[1 - F\left(c_A^{\alpha=1}\right)\right]^{n-v}\frac{1}{v+1} = c_B^{\alpha \to 1}.$$

Because the equilibrium threshold is continuous in α both in a neighborhood of $\alpha = \frac{1}{2}$ and $\alpha = 1$, it follows that $F(c_B^*)$ is higher in a neighborhood of $\alpha = 1$ than at $\alpha = \frac{1}{2}$.